# CS 341: ALGORITHMS

**Lecture 17: max flow**

Readings: CLRS 26.2

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# QUICK REVIEW OF LAST TIME

# RECALL: MAX-FLOW MIN-CUT THEOREM

- **Theorem 3:** every max $s$-$t$ flow has value equal to the capacity of a min $s$-$t$ cut

- We give an **algorithmic proof** of this theorem
  - (showing that one algorithm solves both max-flow and min-cut at the same time)

# FORD-FULKERSON METHOD

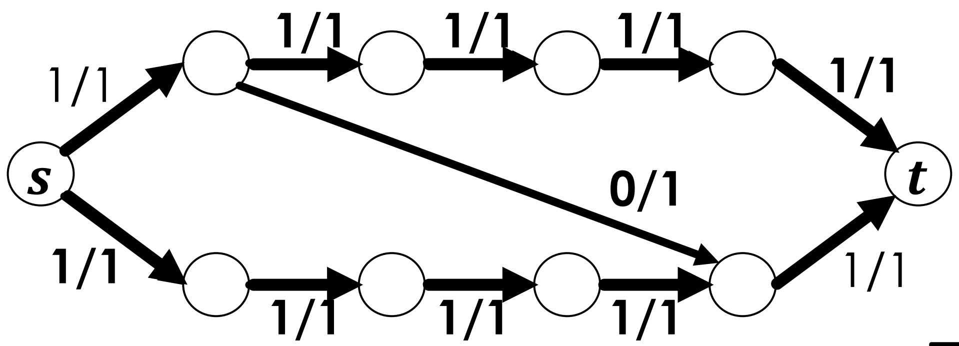
**Algorithm development**

(mixed together with proof of max-flow min-cut theorem)

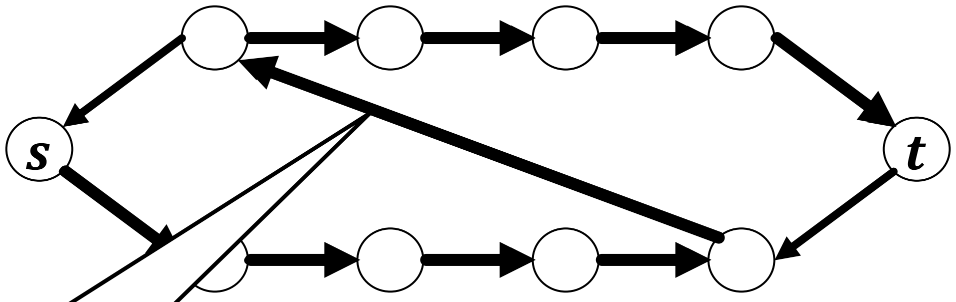
# FORD-FULKERSON METHOD

Same Ford as in
Bellman-Ford :)

- Can **undo** previous decisions to improve the flow
  - Can effectively "push back" some flow
    using an **augmenting path** through a **residual graph**




improved flow



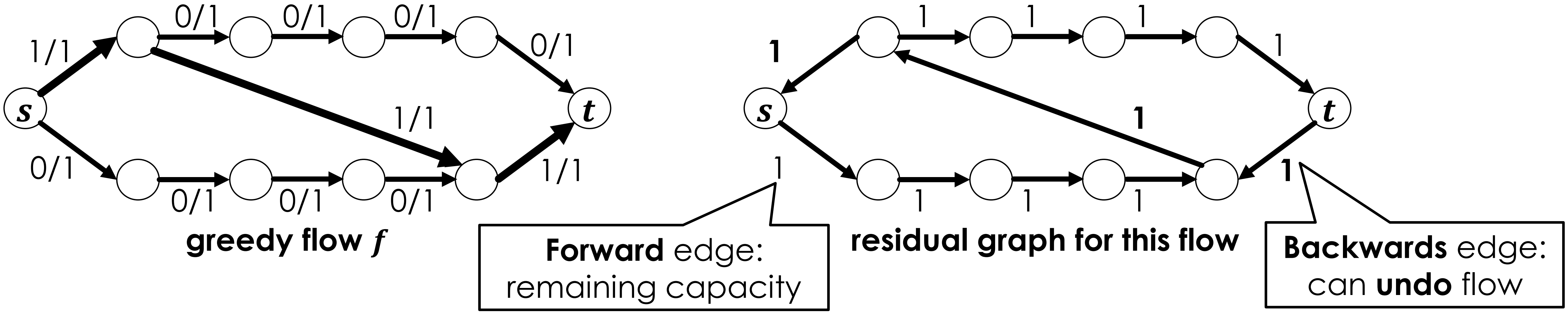

Pushes back the
flow on this edge
(negating its flow)

**"augmenting path"**

So, what's the residual graph,
how do we find an augmenting path,
and how do we improve the flow?

# RESIDUAL GRAPH

- A **residual graph** $R_f$ is defined for a **given flow** $f$ and **graph** $G$
- $R_f$ has the same vertices as $G$
- For each edge $e = uv$ in $G$,
  - If $f(\mathrm{e}) < c(e)$, then $R_f$ contains a **forward** edge $(u, v)$ with the **remaining capacity** $c(e) - f(e)$
  - If $f(e) > 0$, then $R_f$ contains a **backwards** edge $(v, u)$ with **capacity** $f(e)$ representing flow that could be "pushed back"

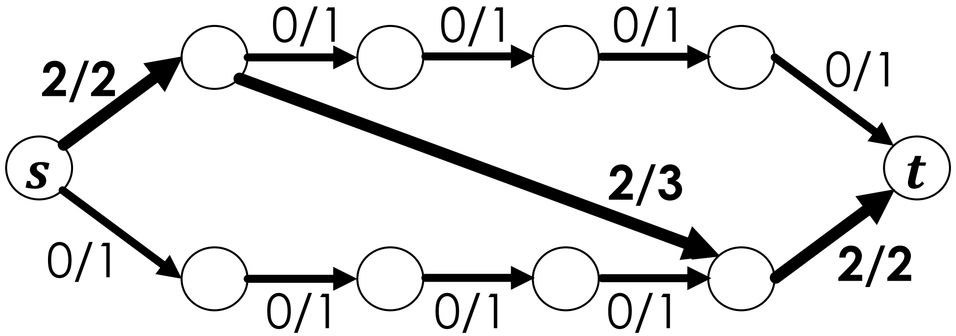



greedy flow $f$

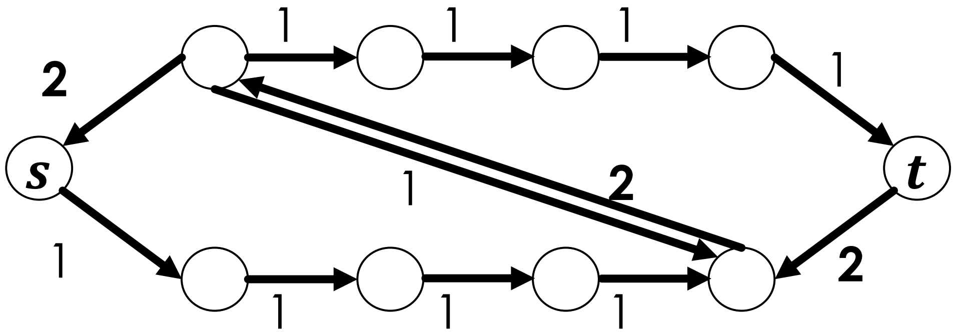
residual graph for this flow

# ANOTHER EXAMPLE RESIDUAL GRAPH

- Recall: for each edge $e = uv$ in $G$,
  - If $f(\mathrm{e}) < c(e)$, then $R_f$ contains a **forward** edge $(u, v)$ with the **remaining capacity** $\boldsymbol{c(e) - f(e)}$
  - If $f(e) > 0$, then $R_f$ contains a **backwards** edge $(\boldsymbol{v}, \boldsymbol{u})$ with **capacity** $\boldsymbol{f(e)}$ representing flow that could be "pushed back"




**flow $f$ with value 2**

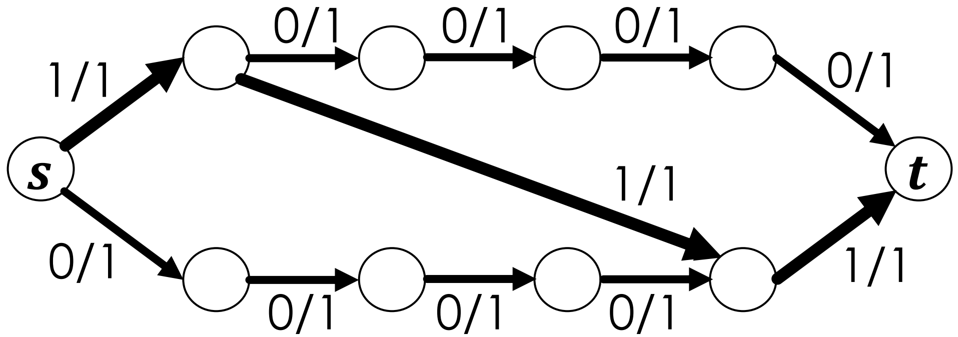



**residual graph for this flow**

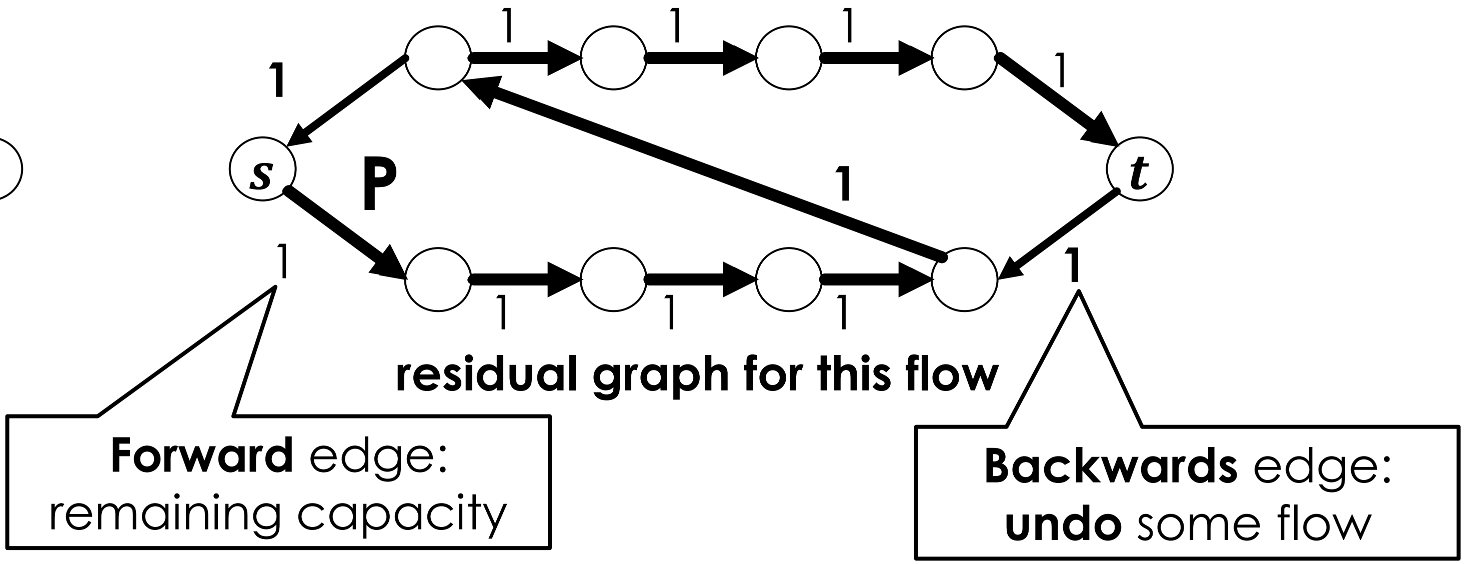
# CONTINUING WITH NEW MATERIAL

# FORD-FULKERSON METHOD

- Find a **shortest path P** from $s$ to $t$ in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
  - And use it to **update the flow**



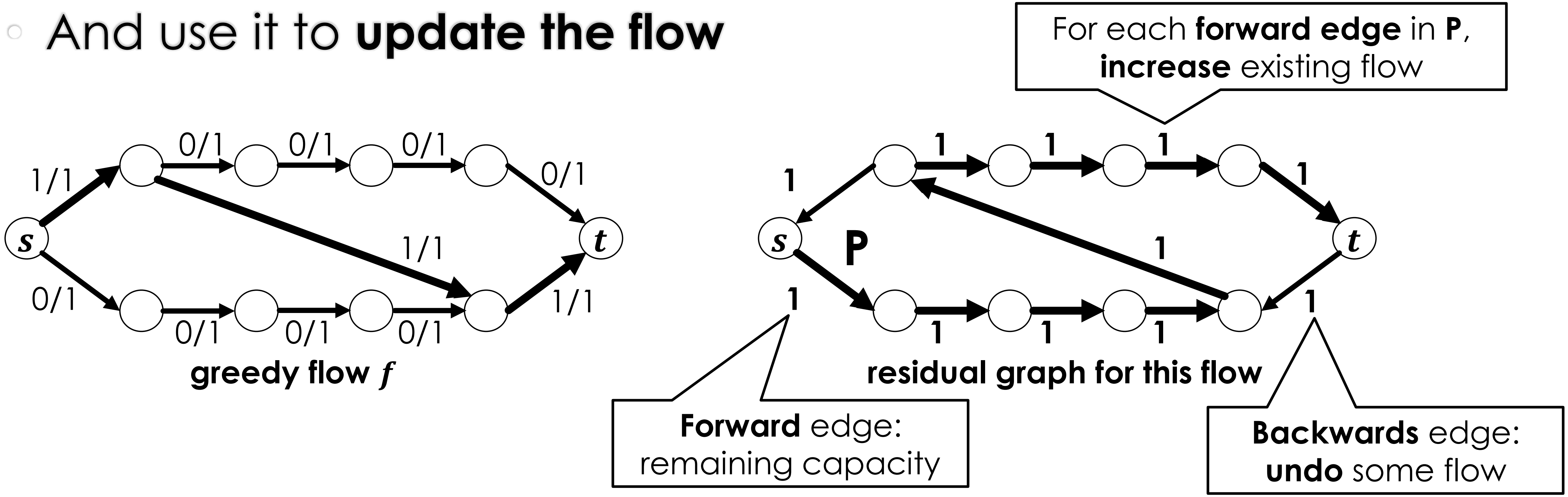

**greedy flow $f$**




**residual graph for this flow**

# FORD-FULKERSON METHOD

- Find a **shortest path P** from $s$ to $t$ in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
  - And use it to **update the flow**



# FORD-FULKERSON METHOD

- Find a **shortest path P** from $s$ to $t$ in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
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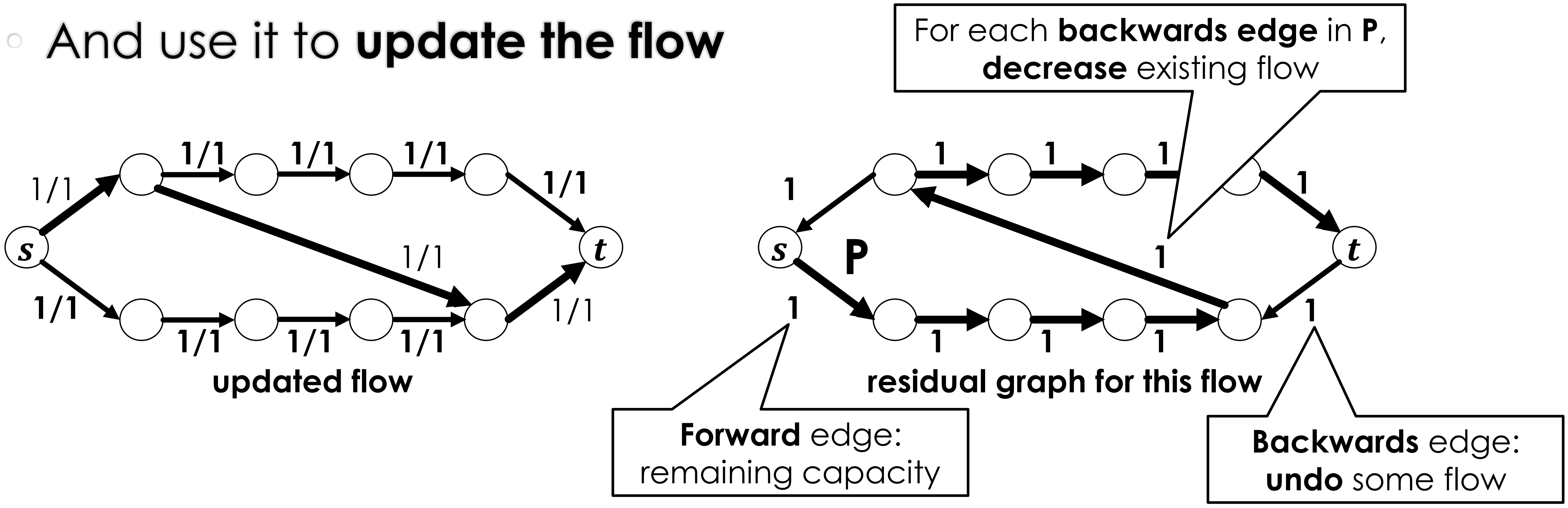
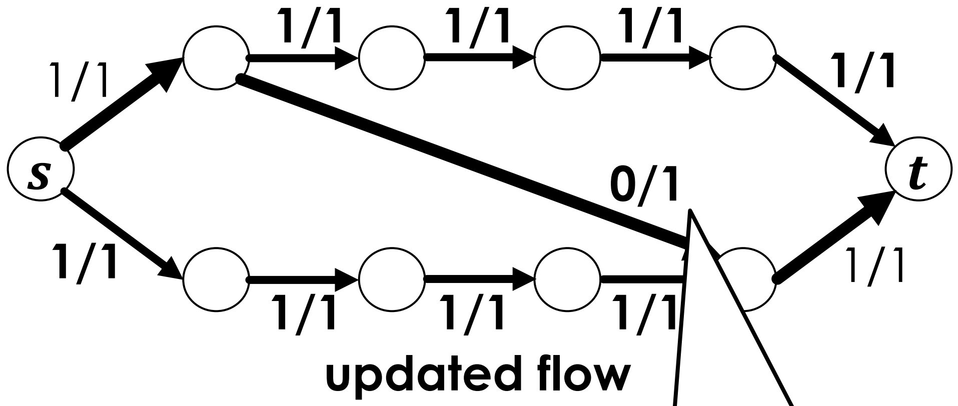
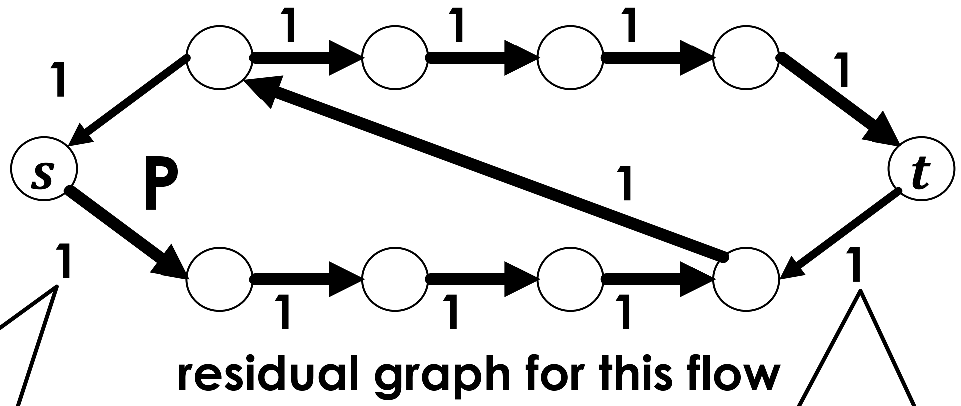


# FORD-FULKERSON METHOD

- Find a **shortest path P** from $s$ to $t$ in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
  - And use it to **update the flow**




Original greedy path no longer exists

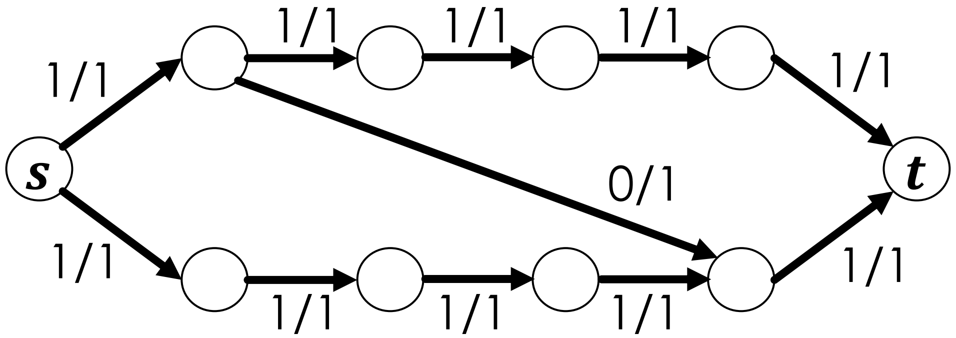



**Forward** edge: remaining capacity

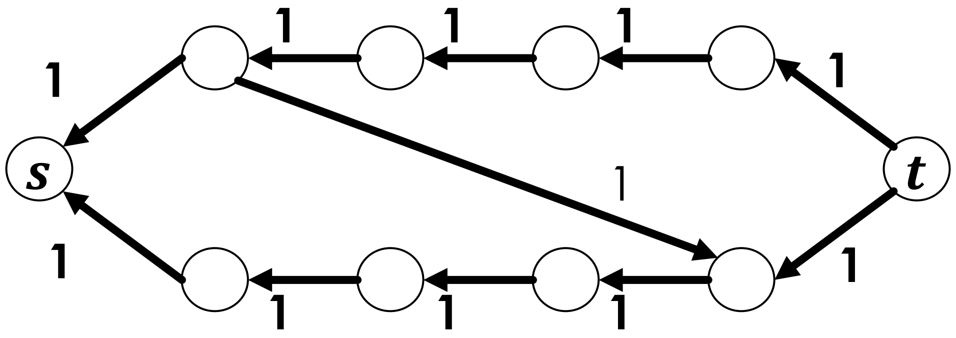
**Backwards** edge: **undo** some flow

# FORD-FULKERSON METHOD

- Find a **shortest path P** from $s$ to $t$ in the **residual graph**
  - If it **improves** the flow, we call it an **augmenting path**
  - And use it to **update the flow**




**updated flow**




**new residual graph for this flow**

No path from $s$ to $t$ in residual graph. Done!

# IMPROVING A FLOW $f$

## GIVEN AN AUGMENTING PATH $P$

no cycles!

- An augmenting path w.r.t a flow $f$ is a **simple** $s$-$t$ path in $R_f$
- Let $P$ be an augmenting path w.r.t $f$
- Let $\text{bottleneck}(f, P)$ be the minimum capacity of an edge in $P$
- We show this subroutine $\text{augment}(f, P)$ always improves the value of flow $f$

```
augment(f, P)
    let b = bottleneck(f, P)
    for each edge e = (u,v) in P
        if e is a forward edge
            f(e) = f(e) + b
        else if e is a backwards edge
            let e' = (v,u)
            f(e') = f(e') - b
```

# LEMMA 4: AUGMENT() IMPROVES FLOW $f$

- Let $f$ be a flow in $G$ with $f^{in}(s) = 0$,
  and $P$ be an augmenting path w.r.t $f$
- Let $f'$ be the resulting flow after running $\text{augment}(f, P)$
- Then $\boldsymbol{f'}$ **is a flow with** $\textbf{value}(\boldsymbol{f'}) = \textbf{value}(\boldsymbol{f}) + \textbf{bottleneck}(\boldsymbol{f}, \boldsymbol{P})$

- That is, $\textbf{augment}(\boldsymbol{f}, \boldsymbol{P})$ **increases the flow by** $\textbf{bottleneck}(\boldsymbol{f}, \boldsymbol{P})$

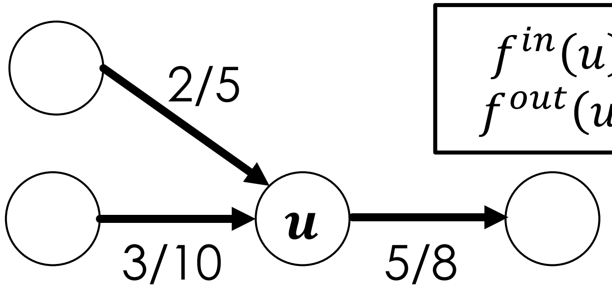
# PROOF

- Claim: $\mathbf{augment}(f, P)$ increases the flow by $\mathbf{bottleneck}(f, P)$
- First check $f'$ is a flow
  - Prove capacity and conservation constraints, and $f'^{in}(s) = 0$
- **Are capacity constraints satisfied?**
  - We add/subtract $\mathbf{bottleneck}(f, P)$ to/from each edge
  - And $\mathbf{bottleneck}(f, P)$ is the minimum of the smallest remaining capacity, and the current flow
  - So capacity constraints are satisfied

# PROOF

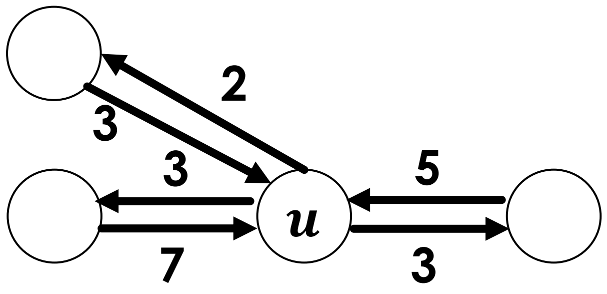
- Claim: $\text{augment}(f, P)$ increases the flow by $\text{bottleneck}(f, P)$
- **How about conservation of flow?**
  - Consider how the flow into and out of each vertex $u \notin \{s, t\}$ changes as a result of running $\text{augment}(f, P)$
  - We show the change in $f^{in}(u)$
    is the same as the change in $f^{out}(u)$
  - There are 4 cases, depending on whether the edges entering/leaving $u$ are **forward** or **backward** edges

## Case 1: forward / forward

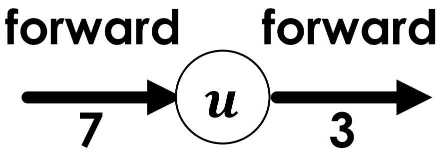



$f^{in}(u) = 5$
$f^{out}(u) = 5$

flow $\boldsymbol{f}$

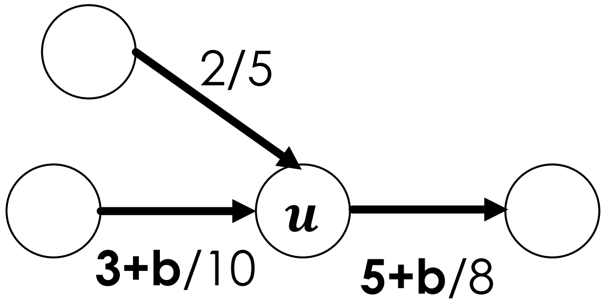



residual graph $\boldsymbol{R_f}$




**augmenting path $\boldsymbol{P}$ in $\boldsymbol{R_f}$**

**Let $\mathbf{bottleneck}(\boldsymbol{f}, \boldsymbol{P}) = \boldsymbol{b}$**



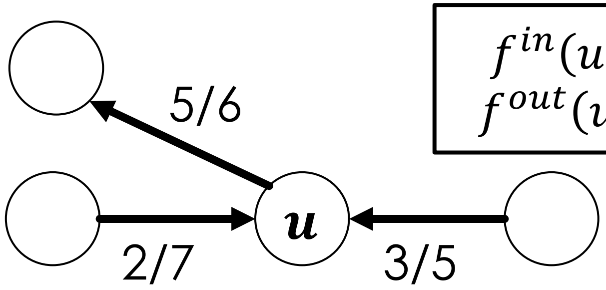

$f'^{in}(u) = 5 + b$
$f'^{out}(u) = 5 + b$

new flow $\boldsymbol{f'}$
(after augmenting)

Both $f^{in}(u)$ and $f^{out}(u)$ are increased by $\text{bottleneck}(f, P)$

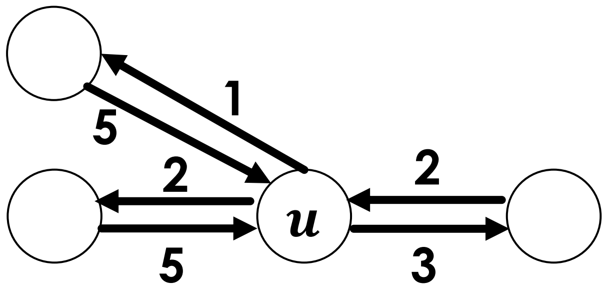
**Case 2: backwards / backwards** is similar.
Both $f^{in}(u)$ and $f^{out}(u)$ are decreased by $\boldsymbol{b}$

## Case 3: forward / backwards

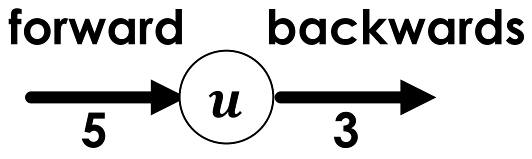



$$f^{in}(u) = 5$$
$$f^{out}(u) = 5$$

flow $\boldsymbol{f}$

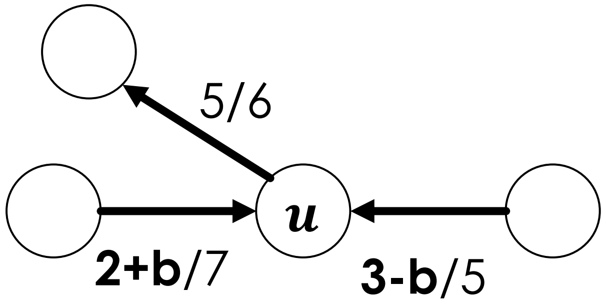



residual graph $\boldsymbol{R_f}$




**augmenting path $\boldsymbol{P}$ in $\boldsymbol{R_f}$**

**Let $\mathbf{bottleneck}(\boldsymbol{f}, \boldsymbol{P}) = \boldsymbol{b}$**




$$f'^{in}(u) = 5$$
$$f'^{out}(u) = 5$$

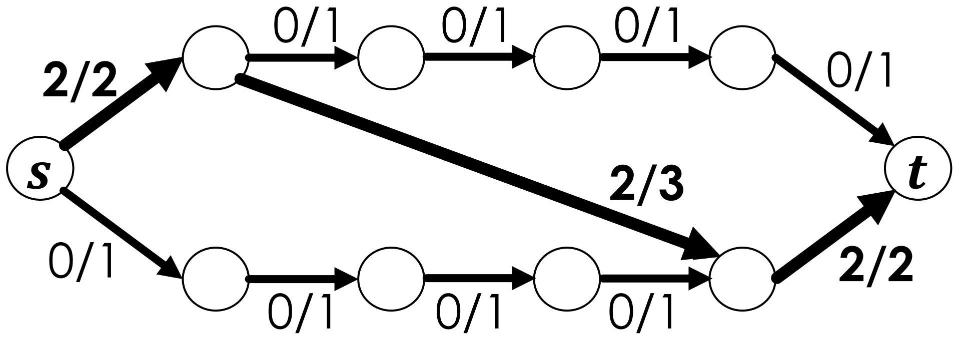
new flow $\boldsymbol{f'}$
(after augmenting)

Added and subtracted $\boldsymbol{b}$ terms **cancel out**

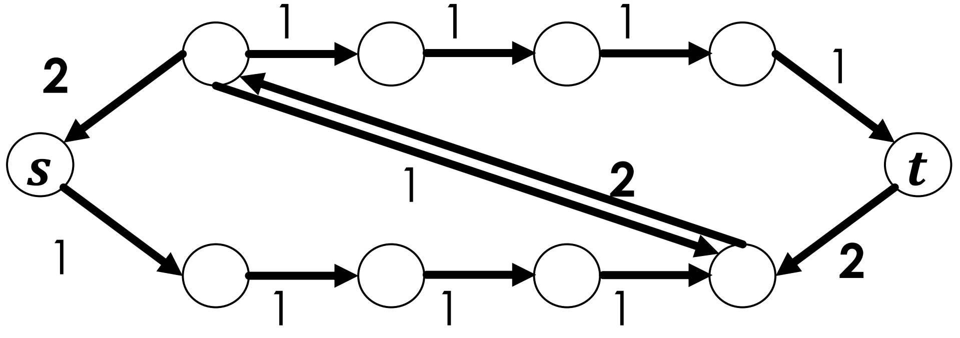
**Case 4: backwards / forwards** is similar.

# SHOWING ${f'}^{in}(s) = 0$

- Last step in showing $f'$ is a flow
  - Prove: $s$ still has no flow into it
- Since $f$ is a flow, $f^{in}(s) = 0$
- To get ${f'}^{in}(s) > 0$, an augmenting path must include an edge **into $s$**
- But then an augmenting path starts at $s$, then returns to $s$, forming a cycle -- contradiction!




**flow $f$ with value 2**




**residual graph for this flow**

# FINISHING LEMMA 4: AUGMENT() IMPROVES FLOW

- Finally we argue $\text{value}(f') = \text{value}(f) + \text{bottleneck}(f, P)$
- $f$ and $f'$ are flows, so $\text{value}(f') = f'^{out}(s)$ and $value(f) = f^{out}(s)$
- We thus show $f'^{out}(s) = f^{out}(s) + \text{bottleneck}(f, P)$
- The augmenting path $P$ is a **simple** path (leaving $s$ exactly once)
- And there is no flow into $s$,
  so the edge $e \in P$ leaving $s$ is a **forward edge**
- This means $\text{augment}(f, P)$ **adds** $\text{bottleneck}(f, P)$ to $f(e)$
- So $f'^{out}(s) = f^{out}(s) + \text{bottleneck}(f, P)$

# FORD-FULKERSON METHOD

- By Lemma 4, starting from any flow $f$,
  if we can **find an augmenting path** $P$ w.r.t $f$ in $R_f$,
  then we can use $\text{augment}(f, P)$ to **improve our flow**
- Ford-Fulkerson does this repeatedly **starting** from an **empty flow**

```
FordFulkerson(G=(V,E))
    for e in E
        f(e) = 0

    while there is a simple s-t path P in Rf do
        augment(f, P)
        and update the residual graph Rf
```

What we have proved so far: **augmenting improves flow.**

We **don't know yet** if

1. **we can actually obtain the max flow**, or
2. **whether max-flow = min-cut.**

# MAX-FLOW MIN-CUT THEOREM PROOF

# PROOF STRATEGY

- Claim: when there is **no augmenting path**, there is a **cut with capacity equal to the value of the current flow**.
- Proving this will simultaneously
  - prove the max-flow min-cut theorem,
  - prove correctness of the Ford-Fulkerson method,
  - solve the max flow problem, and
  - solve the min cut problem

# PROVING MAX FLOW = MIN CUT

Two directions:
**max flow** $\leq$ **min cut** and **max flow** $\geq$ **min cut**

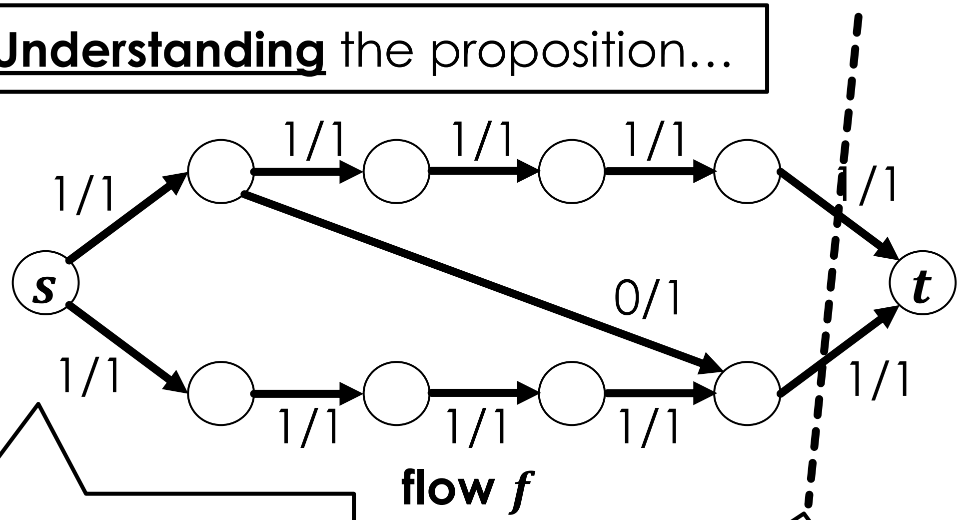
We actually proved the $\leq$ direction already (**Lemma 2 last time**) when discussing upper bounds for max flow

It remains to prove the $\geq$ direction.

# PROVING MAX FLOW ≥ MIN CUT

- Proposition: if $f$ is an $s$-$t$ flow such that
  there is no $s$-$t$ path in the residual graph $R_f$,
  then there is an $s$-$t$ cut $S$ s.t. $\text{value}(f) = c^{out}(S)$

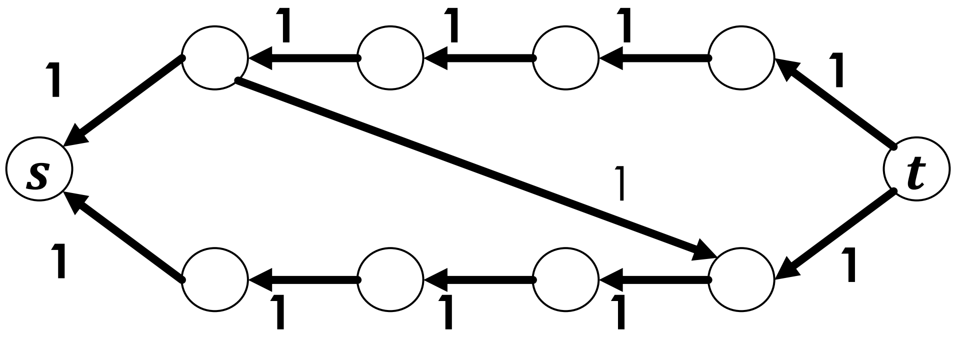
**<u>Understanding</u>** the proposition…




**flow $f$**

If flow value = 2

then cut exists with
capacity 2 = flow value

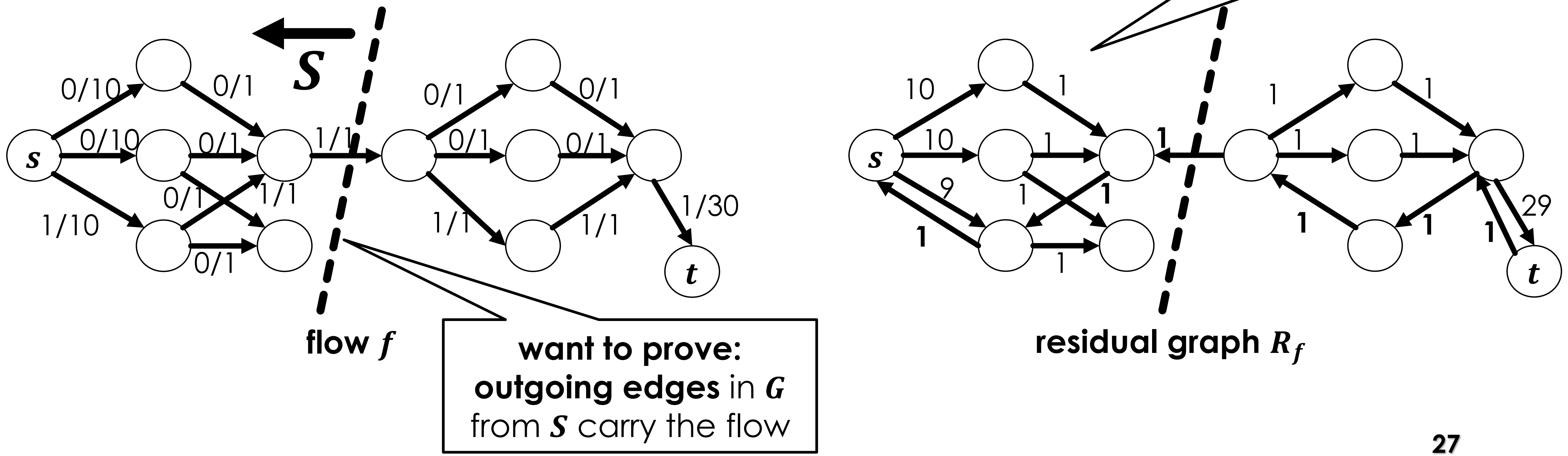



**residual graph $R_f$**
containing **no** $s$-$t$ paths

# PROVING THE PROPOSITION

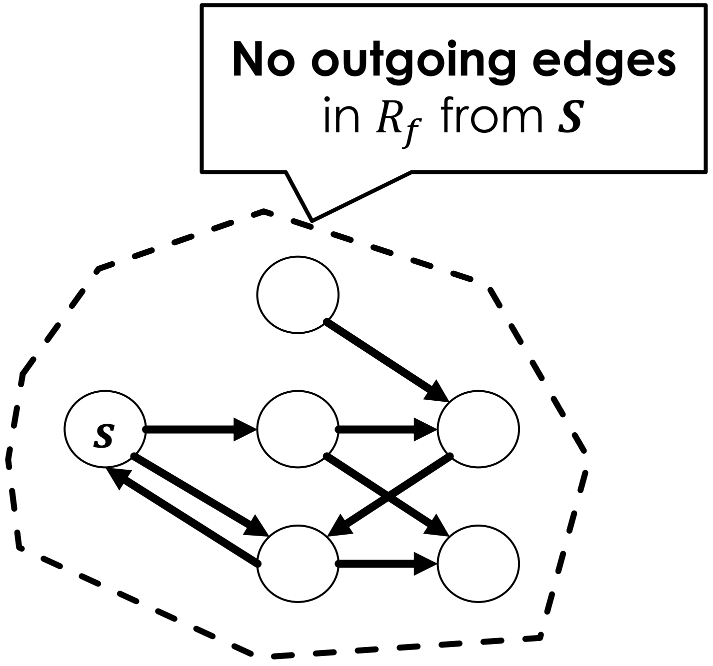
- Since there is no $s$-$t$ path in $R_f$,
  there is a subset $S$ of vertices with $s \in S$, $t \notin S$
  such that $S$ has **no outgoing edges** in $R_f$
- What does this statement look like?

**No outgoing edges** in $R_f$ from $\boldsymbol{S}$

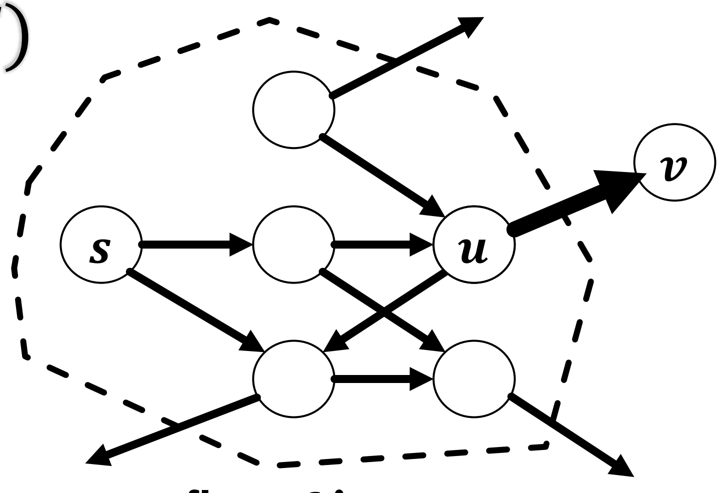


# PROVING THE PROPOSITION

- Since there is no $s$-$t$ path in $R_f$, there is a subset $S$ of vertices with $s \in S$, $t \notin S$ such that $S$ has **no outgoing edges** in $R_f$
- Claim: $\boldsymbol{c^{out}(S)} = \text{value}(f)$
- Consider two types of edges. Type 1:
  - $uv$ **exiting $\boldsymbol{S}$ in $\boldsymbol{G}$** ($uv \in \delta^{out}(S)$ in $G$, $u \in S, v \notin S$)
  - Since $S$ has no outgoing edge in $R_f$, we know $uv \notin R_f$
  - This implies $f(uv) = c(uv)$, as otherwise $uv$ would be a forward edge in $R_f$




**residual graph $R_f$**

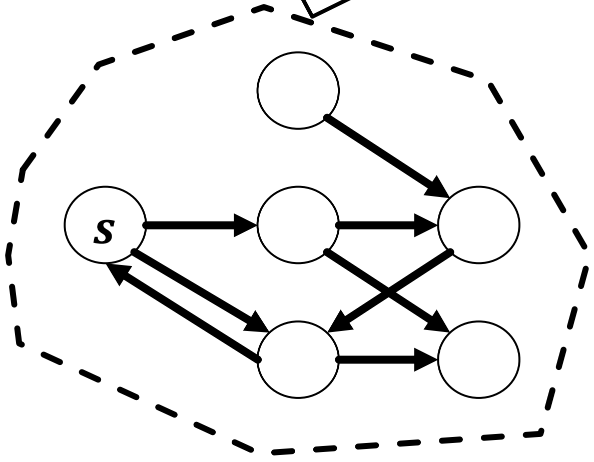



**flow $f$ in $G$**

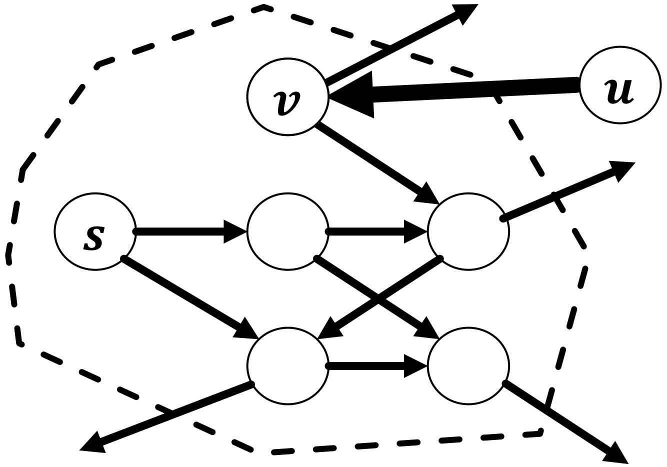
# PROVING THE PROPOSITION

- Claim: $\boldsymbol{c^{out}(S)} = \text{value}(f)$
- Consider two types of edges. Type 2:
  - $uv$ **entering $\boldsymbol{S}$ in $\boldsymbol{G}$**
    ($uv \in \delta^{in}(S)$ in $G$, $u \notin S, v \in S$)
  - Since $S$ has no outgoing edge in $R_f$, we know there is no edge $vu \notin R_f$ (note $vu$ would be directed out of $S$)
  - This implies $f(uv) = 0$, as otherwise $vu$ would be a backwards edge in $R_f$

**No outgoing edges** in $R_f$ from $\boldsymbol{S}$



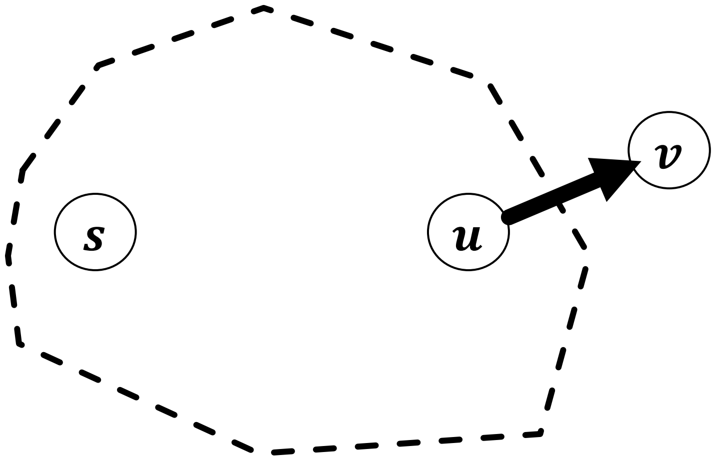

**residual graph $\boldsymbol{R_f}$**

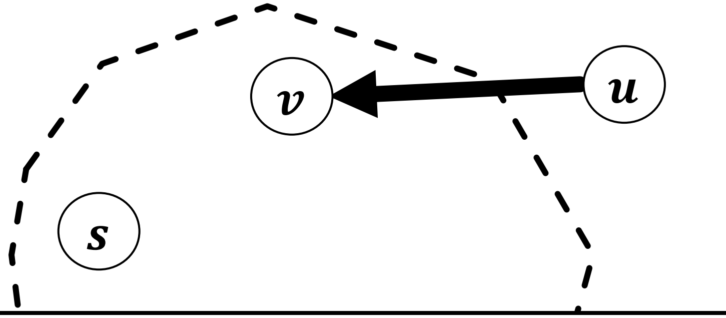



**flow $\boldsymbol{f}$ in $\boldsymbol{G}$**

# PROVING THE PROPOSITION

- We just showed
  - For edge $uv$ directed out of $S$, $f(uv) = c(uv)$
  - For edge $uv$ directed into $S$, $f(uv) = 0$
- So $f^{out}(S) - f^{in}(S) = c^{out}(S) - 0 = c^{out}(S)$
- This proves the proposition. I.e., given flow $f$, if there are no $s$-$t$ paths in $R_f$, then **there is a cut matching the flow**




Note this was the last thing remaining to prove the min-cut max-flow theorem, and the correctness of Ford-Fulkerson

# TIME COMPLEXITY

of the Ford-Fulkerson method

# RUNTIME OF FORD-FULKERSON

- Depends on the implementation

```
FordFulkerson(G=(V,E))
    for e in E
        f(e) = 0

    while there is a simple s-t path P in Rf do
        augment(f, P)
        and update the residual graph Rf
```

- How do we find an augmenting path?
- How many times do we need to augment before we terminate?

# RUNTIME OF FORD-FULKERSON

- Assume we find any arbitrary augmenting path $P$, using any technique, in $O(n+m)$ time
- Then every time $\text{augment}(f, P)$ is run, we know only that the flow **increases**
- If capacities are **integers**, the increase is at least 1
- In this case, **if max flow is $k$ then runtime is $O(k(n+m))$**
  - For max flow we assume a connected graph, so this is $O(km)$
  - **Very bad if $k$ is large**

If capacities are reals (and in particular some are irrational), this may **never** terminate!

# WORST CASE FOR THIS APPROACH

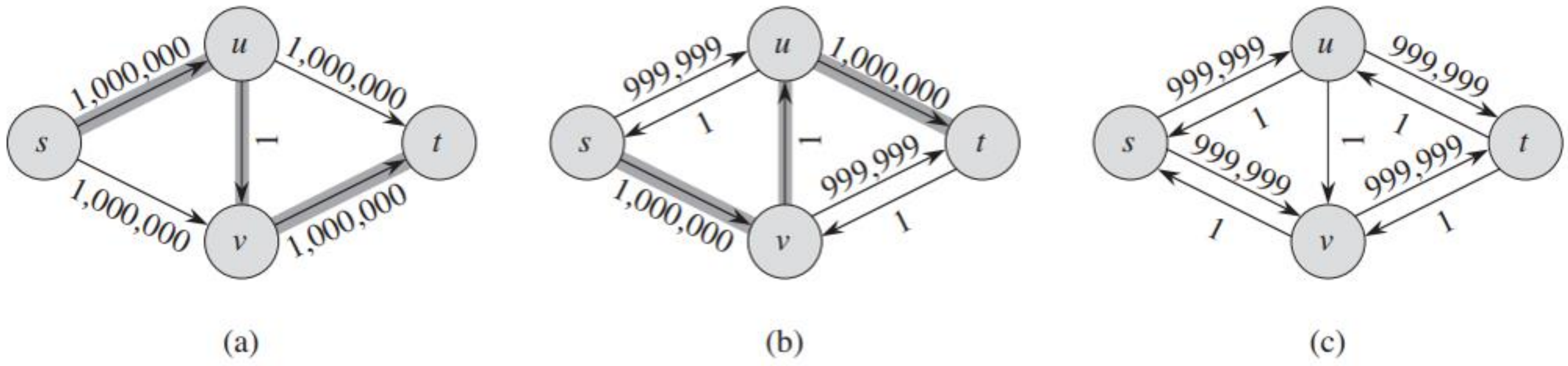



**Figure 26.7** **(a)** A flow network for which FORD-FULKERSON can take $\Theta(E\,|f^*|)$ time, where $f^*$ is a maximum flow, shown here with $|f^*| = 2{,}000{,}000$. The shaded path is an augmenting path with residual capacity 1. **(b)** The resulting residual network, with another augmenting path whose residual capacity is 1. **(c)** The resulting residual network.

# EDMONDS-KARP APPROACH

- **Use BFS** to find a shortest path (in terms of number of edges) and use that as an augmenting path
- It turns out this always **terminates after $O(nm)$ augmenting paths**
  - (even with real capacities)
- BFS takes $O(n+m)$ time; $\boldsymbol{O(m)}$ since the graph is connected
- **So total runtime is $O(nm^2)$**

There are more sophisticated algorithms with $O(V^2E)$ and even $O(V^3)$ runtimes (**optional**: CLRS 26.4, 26.5)

In 2022, researchers found <u>an *almost* linear time algorithm</u>, which leverages techniques from convex optimization and sophisticated data structures