CS 341: ALGORITHMS

Lecture 20: intractability II – complexity class NP

Readings: see website

Trevor Brown

https://student.cs.uwaterloo.ca/~cs341

trevor.brown@uwaterloo.ca

THIS TIME

- **Finishing TSP reductions**
- Complexity class **NP**
  - Oracles, certificates, polytime verification algorithms

RECALL

- So far we know
  - TSP-Dec $\leq_P^T$ TSP-Optimal Value
  - TSP-Dec $\leq_P^T$ TSP-Optimization
- **In progress**
  - **TSP-Optimal Value $\leq_P^T$ TSP-Dec**

Travelling Salesperson Problems

Problem 7.5

TSP-Optimization

Instance: *A graph G and edge weights $w : E \to \mathbb{Z}^+$.*

Find: *A hamiltonian cycle H in G such that $w(H) = \sum_{e \in H} w(e)$ is minimized.*

Problem 7.6

TSP-Optimal Value

Instance: *A graph G and edge weights $w : E \to \mathbb{Z}^+$.*

Find: *The minimum T such that there exists a hamiltonian cycle H in G with $w(H) = T$.*

Problem 7.7

TSP-Decision

Instance: *A graph G, edge weights $w : E \to \mathbb{Z}^+$, and a target T.*

Question: *Does there exist a hamiltonian cycle H in G with $w(H) \leq T$?*

What's the size of the input $I = (G, w)$?

$Size(I) = Size(G) + Size(w)$

So, suppose G is represented as an **array of adjacency lists** (one list for each vertex), with each list containing **edges** to neighbouring vertices, and an edge is represented by a **weight** and the **name** of the target vertex

Bits to store **weight** of the edge (storing $w(e)$ takes $\log w(e) + 1$ bits)

Bits to store the **name** of the target vertex (in 1..|V|)

$$Size(I) = |V| + \sum_{e \in E} (\log w(e) + 1 + \log|V| + 1)$$

Array of empty lists for all vertices v

For all edges

Let's relate this to runtime… what's the runtime?

TSP-Optimal Value $\leq_P^T$ TSP-Dec

Let's assume $O(1)$ time for operations on weights	Technically not needed to show polytime.. But simplifies

Algorithm: *TSP-OptimalValue-Solver*(G, w)
external *TSP-Dec-Solver*
$hi \leftarrow \sum_{e \in E} w(e)$ — $O(|E|)$
$lo \leftarrow 0$ — $O(1)$
if not *TSP-Dec-Solver*(G, w, hi) **then return** (∞) — $O(1)$ for the oracle
while $hi > lo$ — # iterations: $O(\log(hi - lo))$ $= \log \sum_{e \in E} w(e)$
do
- $mid \leftarrow \left\lfloor \frac{hi+lo}{2} \right\rfloor$
- **if** *TSP-Dec-Solver*(G, w, mid)
- **then** $hi \leftarrow mid$
- **else** $lo \leftarrow mid + 1$

(loop body: $O(1)$)

return (hi)

Runtime $T(I) \in O(|E| + \log \sum_{e \in E} w(e))$

COMPARING $T(I)$ AND $Size(I)$

- $T(I) \in O(|E| + \log \sum_{e \in E} w(e))$
- $Size(I) = |V| + \sum_{e \in E}(\log w(e) + 1 + \log|V| + 1)$

$$= |V| + \Sigma_{e \in E}(\log w(e) + 1) + \boldsymbol{\Sigma_{e \in E}}(\log|V| + 1)$$

$$= |V| + \Sigma_{e \in E}(\log w(e) + 1) + \Sigma_{e \in E}(\log|V|) + |\boldsymbol{E}|$$

- Want to show $T(I) \in O(Size(I)^c)$ for some constant c (we show c=1)

$$O(|\boldsymbol{E}| + \log \sum_{e \in E} w(e)) \subseteq^? O(|V| + \Sigma_{e \in E}(\log w(e) + 1) + \Sigma_{e \in E} \log|V| + |\boldsymbol{E}|)$$

$$\Leftrightarrow O(\log \sum_{e \in E} w(e)) \subseteq^? O(|V| + \Sigma_{e \in E}(\log w(e) + 1) + \Sigma_{e \in E} \log|V|)$$

How to compare $\boldsymbol{\log \sum_{e \in E} w(e)}$ and $\boldsymbol{\Sigma_{e \in E}(\log w(e) + 1)}$?

COMPARING $T(I)$ AND $Size(I)$

- **How to compare $\mathbf{\log \sum_{e \in E} w(e)}$ and $\mathbf{\Sigma_{e \in E}(\log w(e) + 1)}$?**
- $\mathbf{\Sigma_{e \in E}(\log w(e) + 1)} = (\log w(e_1) + 1) + (\log w(e_2) + 1) + \cdots + \left(\log\left(w\left(e_{|E|}\right)\right) + 1\right)$
- Can we combine these terms into one log using $\log x + \log y = \log xy$?
- $\mathbf{\Sigma_{e \in E}(\log w(e) + 1)} = (\log w(e_1) + \log 2) + + \cdots + \left(\log\left(w\left(e_{|E|}\right)\right) + \log 2\right)$
- $\mathbf{\Sigma_{e \in E}(\log w(e) + 1)} = \log 2w(e_1)\, 2w(e_2) \ \ldots\ 2w\left(e_{|E|}\right) = \mathbf{\log \prod_{e \in E} 2w(e)}$
- So how to compare $\mathbf{\log \prod_{e \in E} 2w(e)}$ and $\mathbf{\log \sum_{e \in E} w(e)}$**?**
  - **All $\mathbf{w(e)}$** are positive integers, so $\mathbf{\prod_{e \in E} 2w(e) \geq \sum_{e \in E} w(e)}$
  - Since log is increasing on $\mathbb{Z}^+$, $\mathbf{\log \prod_{e \in E} 2w(e) \geq \log \sum_{e \in E} w(e)}$

COMPARING $T(I)$ AND $Size(I)$

- We in fact show $\boldsymbol{T(I)} \in O(Size(I))$

$$O(\log \sum_{e\in E} w(e)) \subseteq^? O(|V| + \Sigma_{e\in E}(\log w(e) + 1) + \Sigma_{e\in E} \log|V|)$$

How to compare $\log \sum_{e\in E} w(e)$ and $\boldsymbol{\Sigma_{e\in E}(\log w(e) + 1)}$?

We just saw $\boldsymbol{\Sigma_{e\in E}(\log w(e) + 1) = \log \prod_{e\in E} 2w(e) \geq \log \sum_{e\in E} w(e)}$

So $T(I) \in O(Size(I)^c)$ where $c = 1$

So this reduction has runtime that is polynomial in the input size!

TSP-Optimal Value $\leq_P^T$ TSP-Dec

So TSP-OptimalValue-Solver is polytime… But is it a **correct reduction from TSP-Optimal Value to TSP-Dec?**

Algorithm: *TSP-OptimalValue-Solver*(G, w)
external *TSP-Dec-Solver*
$hi \leftarrow \sum_{e \in E} w(e)$
$lo \leftarrow 0$
if not *TSP-Dec-Solver*(G, w, hi) **then return** (∞)
while $hi > lo$
do $\begin{cases} mid \leftarrow \lfloor \frac{hi+lo}{2} \rfloor \\ \textbf{if } \textit{TSP-Dec-Solver}(G, w, mid) \\ \quad \textbf{then } hi \leftarrow mid \\ \quad \textbf{else } lo \leftarrow mid + 1 \end{cases}$
return (hi)

Need to prove:
TSP-OptimalValue-Solver(G,w)
returns the weight $\boldsymbol{W}$
of the **shortest Hamiltonian Cycle (HC) in G**

Sketch: We return ∞ iff there is **no HC**.
Key loop invariant: $\boldsymbol{W} \in [\boldsymbol{lo}, \boldsymbol{hi}]$.
So, at termination when $hi = lo$,
we return exactly $hi = W$.

TSP-Optimal Value $\leq_P^T$ TSP-Dec

Algorithm: *TSP-OptimalValue-Solver*(G, w)
external *TSP-Dec-Solver*
$hi \leftarrow \sum_{e \in E} w(e)$
$lo \leftarrow 0$
if not *TSP-Dec-Solver*(G, w, hi) **then return** (∞)
while $hi > lo$
do
- $mid \leftarrow \lfloor \frac{hi+lo}{2} \rfloor$
- **if** *TSP-Dec-Solver*(G, w, mid)
  - **then** $hi \leftarrow mid$
  - **else** $lo \leftarrow mid + 1$

return (hi)

So, TSP-OptimalValue-Solver is **polytime**, and is a **correct** reduction.

We have therefore shown: **TSP-Optimal Value is polytime reducible to TSP-Dec**

So, if an $\boldsymbol{O(1)}$ implementation of TSP-Dec-Solver exists, then we have a **polytime** implementation of TSP-Optimal-Value-Solver!

In fact, TSP-OptimalValue-Solver remains **polytime** even if the implementation of the **oracle runs in polytime** instead of O(1)! (bonus slides)

PROVING REDUCTIONS CORRECT

- **In more complex reductions** where we **transform the input** before calling the oracle, we will need a **more complex proof**:
- (A) If there is a(n optimal) solution in the input, our transformation will preserve that solution so the oracle can find it, and
- (B) Our transformation doesn't introduce new solutions that are ***not*** present in the original input
  - *(i.e., if we find a solution in the transformed input, there was a corresponding solution in the original input)*

More on this later…

INPUT SIZE CHEAT SHEET

Input I	**Perfectly fine** choices of $Size(I)$
int x	1 or $\lfloor \log(x) \rfloor + 1$ **(can simplify to $\log(x) + 1$ or $\log x$)**
Graph (V, E) with weights W:	$\|V\|$ or $\|E\|$ or $\|V\|^2$ or $\|V\| + \|E\|$ or $\sum_{e \in E}(\log(w(e)) + 1)$ or $\sum_{u,v \in V}(\log(w(u,v)) + 1)$ or **any sum of terms above**
$A[1..n]$ of int	n or $\sum_i(\log(A[i]) + 1)$
n x n matrix m	n^2 or $\sum_{i,j}(\log(m_{ij}) + 1)$

To write down x=1, need log(1)+1=1 bit. For x=2 this is 2 bits. For x=4, 3 bits.

Pick any expression that makes your analysis easy

Exponentially larger than optimal representation!

Input I	**Examples of BAD** choices of $Size(I)$
int x	x
Graph (V, E)	$2^{\|V\|}$ or $\|V\|^{\|E\|}$ or $\sum_{e \in E} w(e)$
$A[1..n]$ of int	2^n or $\sum_i A[i]$

Pseudo-polynomial ~= no exponentiation of non-constant terms

Technically any **pseudo-polynomial combination** of these terms is fine. For example, the following is fine:
$(|E|^{100} + |V|^2) \cdot \sum_{e \in E}(\log(w(e)) + 1)$

- So far we know
  - TSP-Dec $\leq_P^T$ TSP-Optimal Value
  - TSP-Dec $\leq_P^T$ TSP-Optimization
  - TSP-Optimal Value $\leq_P^T$ TSP-Dec
- Let's show
  - TSP-Optimization $\leq_P^T$ TSP-Dec

WHAT ABOUT REDUCING **TSP-OPTIMIZATION** TO TSP-DEC?

Problem 7.7

TSP-Decision

Instance: *A graph G, edge weights $w : E \to \mathbb{Z}^+$, and a target T.*

Question: ~~*Does there exist*~~ *a hamiltonian cycle H in G with $w(H) \leq T$?*

Need to return the **actual** minimum Hamiltonian Cycle!

Problem 7.7

TSP-Decision

Instance: *A graph G, edge weights $w : E \to \mathbb{Z}^+$, and a target T.*

Question: *Does there exist a hamiltonian cycle H in G with $w(H) \leq T$?*

Given only a **single bit** of information **per call** to the oracle

We already know how to get the **weight** T^* of the minimum HC…

Idea: Use T^* along with calls to the oracle to somehow figure out **which edges** are involved in the minimum HC?

TSP-Optimization $\leq_P^T$ TSP-Dec

Algorithm: *TSP-Optimization-Solver*$(G = (V, E), w)$
external *TSP-OptimalValue-Solver*, *TSP-Dec-Solver*
$T^* \leftarrow$ *TSP-OptimalValue-Solver*(G, w)
if $T^* = \infty$ **then return** ("no hamiltonian cycle exists")
$w_0 \leftarrow w$
$H \leftarrow \emptyset$
for all $e \in E$
do $\begin{cases} w_0[e] \leftarrow \infty \\ \textbf{if not } \textit{TSP-Dec-Solver}(G, w_0, T^*) \\ \quad \textbf{then } \begin{cases} w_0[e] \leftarrow w[e] \\ H \leftarrow H \cup \{e\} \end{cases} \end{cases}$
return (H)

To remove any dependence on this "other oracle," simply replace this call with the reduction **code** we showed

Already know this call is poly-time reducible to TSP-Dec!

If removing edge e removes **every** Hamiltonian cycle of minimum weight

then e is part of **every** minimum Hamiltonian cycle, and we add it to H (and add it back into the graph)

At the end, the graph contains precisely the edges that are needed to produce a minimum HC

[Correctness] **Loop invariant:** there exists a HC of weight T^* in w_0

By the end of the loop, H contains all finite edges in w_0

So some HC $\boldsymbol{C}$ of weight T^* **is contained in $\boldsymbol{H}$**

At the end of the algorithm, there is
a Hamiltonian Cycle $\boldsymbol{C}$ of optimal weight T^* **contained in** H

If $\boldsymbol{H}$ **is precisely** $\boldsymbol{C}$, then we are done.
Suppose not to obtain a contradiction.

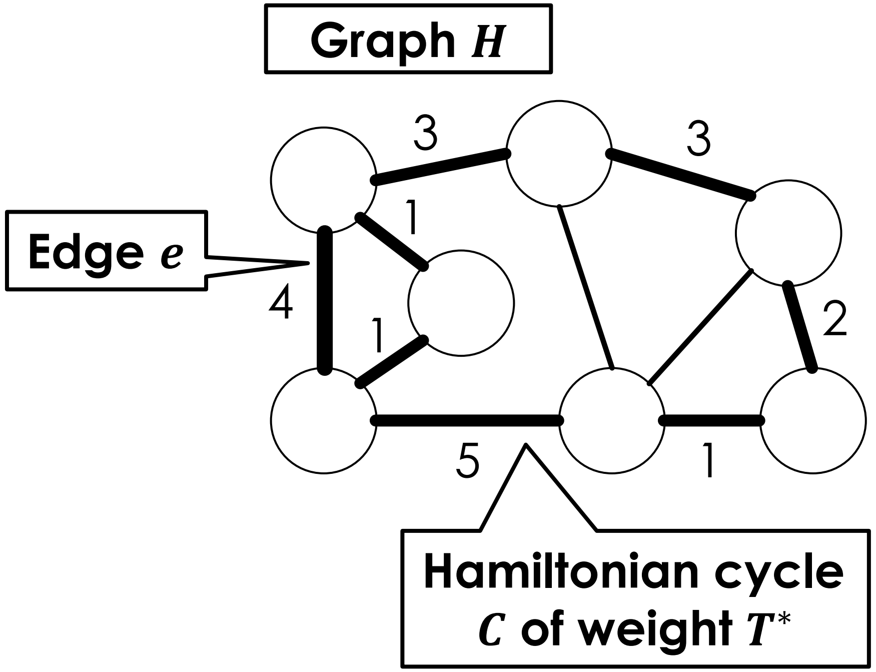
In this case, there are some **other edges** in H as well.

Let e be one such edge.

Consider the iteration when e was processed.
Note e was **not removed** in this iteration!

Doing so would remove **all** Hamiltonian Cycles of weight T^*,
including $\boldsymbol{C}$.

This means the edge must be part of $\boldsymbol{C}$---contradiction!



TSP-Optimization $\leq_P^T$ TSP-Dec

Algorithm: *TSP-Optimization-Solver*$(G = (V, E), w)$
external *TSP-OptimalValue-Solver*, *TSP-Dec-Solver*
$T^* \leftarrow$ *TSP-OptimalValue-Solver*(G, w) — $poly(Size(I))$
if $T^* = \infty$ **then return** ("no hamiltonian cycle exists") — $O(1)$
$w_0 \leftarrow w$ — $O(m)$ to copy matrix
$H \leftarrow \emptyset$ — $O(1)$ to create list
for all $e \in E$ — $O(m)$ iterations
do { $w_0[e] \leftarrow \infty$; **if not** *TSP-Dec-Solver*(G, w_0, T^*) **then** { $w_0[e] \leftarrow w[e]$; $H \leftarrow H \cup \{e\}$ } } — $O(1)$ per iteration
return (H)

So this is a **correct** reduction.
Is it a **polytime reduction**?

What's the runtime?

Let's assume unit costs for simplicity

Runtime = $poly(Size(I)) + O(m)$

What's $Size(I)$?
(What's a "useful" lower bound?)

$Size(I) = \Omega(|E|) = \Omega(m)$

Clearly $O(m) \in O(Size(I)^1)$
So runtime is in $poly(Size(I))$

So **yes**, this is a **polytime reduction**

What would change if we precisely counted the number of bits in each edge, weight, etc., in $Size(I)$?

What if **operations on weight w** took $O(\log w)$ **time**? (bonus slides)

RECAP

- Showed three flavours of TSP are **polytime-equivalent**
  (i.e., if you can solve one flavour in polytime,
  you can solve all three flavours in polytime)
  - One of these was a decision problem (yes/no),
    and the other two were not (total weight, actual cycle)
- **Decision and non-decision flavours**
  of a problem are often polytime-equivalent
- Proofs for a **<u>polytime Turing reduction</u>**
  - **Correctness** (return value is correct for every possible input)
  - **Polytime** (runtime is polynomial in the input size)
    [or poly(some lower bound on the input size)]

Note: only one of my sections got here

COMPLEXITY CLASS **NP**

NP: Non-deterministic polynomial time

EXAMPLE: SUBSET-SUM PROBLEM

- Suppose we are given some integers, -7, -3, -2, 5, 8
- Does **some** subset of these **sum to zero?**
  - In this case, yes: (-3) + (-2) + 5 = 0

Finding such a subset can be extremely difficult

Suppose I give you a **certificate** consisting of an array of numbers, and **claim** it represents such a subset

If I'm telling the truth, then we call this a **yes-certificate.** It is is essentially a **proof** that "yes" is the correct output.

Can you use a yes-certificate to solve the problem efficiently?

Of course, I might lie and give you a subset that does **not sum to zero**…

I could even give you numbers that are **not in the input**…

Can you determine whether I am lying in polynomial time?

SUBSET-SUM VIA NON-DETERMINISTIC ORACLE

- Suppose there is a **non-deterministic oracle**, which returns **a subset that sums to 0 <u>if one exists</u>** and otherwise can return **anything** (even garbage)
- We call the oracle's output a **certificate**
- Given a **certificate**, can you **verify in polytime** whether it describes a solution to the problem?

```
SubsetSumWithOracle(I)
    C = Oracle(I)
    return verify(I, C)

verify(I, C)
    if C not subset of I then return false
    return (sum(C) == 0)
```

Given such an oracle, this algorithm would **solve** subset-sum

If there **exists** a subset that sums to 0, then **C** is one such subset, and we return **true**

Otherwise, either C is not a subset of the input (return false), or C sums to a non-zero value (return false)

"Non-deterministic" is the **N** in **NP**, and it is so named because of oracles

Here "**non-deterministic**" just means the oracle is magically guaranteed to return a yes-certificate if one exists

BONUS SLIDES

TSP-Optimal Value $\le_P^T$ TSP-Dec

TSP-OptimalValue-Solver remains **polytime** even if the **oracle runs in polytime** instead of O(1)!

Algorithm: *TSP-OptimalValue-Solver*(G, w)
external *TSP-Dec-Solver*
$hi \leftarrow \sum_{e \in E} w(e)$
$lo \leftarrow 0$
if not *TSP-Dec-Solver*(G, w, hi) **then return** (∞)
while $hi > lo$
do $\begin{cases} mid \leftarrow \lfloor \frac{hi+lo}{2} \rfloor \\ \textbf{if } \textit{TSP-Dec-Solver}(G, w, mid) \\ \quad \textbf{then } hi \leftarrow mid \\ \quad \textbf{else } lo \leftarrow mid + 1 \end{cases}$
return (hi)

The key idea is: Consider polynomials $P_R(s)$ and $P_O(s)$ representing the runtime of a reduction and its oracle, respectively, on an input of size s.
Worst possible runtime happens if **every step** in the reduction is a call to the oracle.
This is $P_R(s)P_O(s)$ --- multiplication of polynomials.

But **multiplying polynomials** of degrees d_1, d_2 **results in a polynomial** of degree $\le d_1 + d_2$. **Example:**

$$P_1(x) = 5x^2 + 10x + 100$$
$$P_2(x) = 20x^3 + 20$$
$$P_1(x)P_2(x) = (5x^2 + 10x + 100)(20x^3 + 20)$$
$$= 100x^5 + 200x^4 + 2000x^3 + 100x^2 + 200x + 2000$$

Let's assume $O(\log w)$ time for reading/writing/arithmetic operations on each weight w (and $O(\log w)$ space).

Algorithm: *TSP-Optimization-Solver*$(G = (V, E), w)$
external *TSP-OptimalValue-Solver*, *TSP*...
$T^* \leftarrow$ *TSP-OptimalValue-Solver*(G, w) — **Suppose we show** this is $poly(Size(I))$
if $T^* = \infty$ **then return** ("no hamiltonian cycle exists") — $O(1)$
$w_0 \leftarrow w$ — $O(\sum_{u,v \in V} \log w(u,v))$ to copy matrix
$H \leftarrow \emptyset$ — $O(1)$ to create list
for all $e \in E$ — $O(m)$ iterations: **for all** $\boldsymbol{u, v}$
do
- $w_0[e] \leftarrow \infty$ — $O(\log w(u,v))$
- **if not** *TSP-Dec-Solver*(G, w_0, T^*) — $O(1)$
- **then** $w_0[e] \leftarrow w[e]$ — $O(\log w(u,v))$
- $H \leftarrow H \cup \{e\}$ — $O(1)$

return (H)

So this is a **correct** reduction.
Is it a **polytime reduction**?

What's the runtime on such an input?

Runtime = $poly(Size(I))$
$+O(m + \sum_{u,v \in V} \log w(u,v))$

What's $Size(I)$?
(or a useful lower bound on it)

$Size(I) = O(|E| + \sum_{u,v \in V} \log w(u,v))$

Clearly $O(m + \sum_{u,v \in V} \log w(u,v)) \in poly(Size(I))$

So, this is ***still*** a **polytime reduction**

Unit cost vs non-unit cost assumptions usually do **not** usually make a difference…

This should not be surprising, since the same $O(\log w)$ terms are introduced into both space and time complexities…