CS 341: ALGORITHMS

Lecture 24: intractability VI – Decidability, more NPC transformations

Readings: see website

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COMPLEXITY CLASS **EXPTIME**

A very brief overview

(Non-core material)

EXPTIME is the set of all **decision problems** that can be solved in **exponential time**. I.e., in time $O(2^{poly(n)})$ where $poly(n)$ is a polynomial in the input size.

Observe that $\boldsymbol{NP \subseteq EXPTIME}$

The idea is to generate all possible certificates of an appropriate length and check them for correctness using the given certificate verification algorithm. An an example, for **Hamiltonian Cycle**, we could generate all $n!$ certificates and check each one in turn.

$O(n!) \subseteq O(n^n) = O\left(2^{n \log n}\right)$ time

We do not know if there are problems in **NP** that cannot be solved in polynomial time (because the **P** = **NP**? conjecture is not yet resolved). However, it is possible to prove that there exist problems in **EXPTIME** \ **P**.

One such problem is the **Bounded Halting** problem. Here an instance $I = (A, x, t)$, where A is a program, x is an input to A, and t is a positive integer (in binary). The question to be solved is if $A(x)$ halts after at most t computation steps.

The **Bounded Halting** problem can be solved in time $O(t)$, but this is not a polynomial time algorithm because size$(I) = |A| + |x| + \log_2 t$.

Actually, it can be proven that **Bounded Halting** is EXPTIME-complete. This implies that it is in **EXPTIME** $\setminus$ **P**, since it is known that **EXPTIME** $\neq$ **P**.

t is exponential in $\log t$.
(And $\log t$ might be the largest term in the input size,
in which case $O(t)$ would be **exponential in the input size**.)

UNDECIDABILITY

Problems that are ***impossible*** to solve

DECIDABLE VS UNDECIDABLE PROBLEMS

We say an algorithm $\boldsymbol{A}$ "solves" a decision problem if, for ***every*** instance I, $\boldsymbol{A}(\boldsymbol{I})$ has **finite** runtime and returns the correct answer

If an algorithm A **solves** decision problem Π, then we say Π is **decidable**.

Formally, Π is **decidable IFF** there exists some algorithm A such that, for **every** instance I, $A(I)$ returns the correct answer in **finite** time.

If it is ***not possible*** to design an algorithm A that **solves** decision problem Π, then we say Π is **undecidable**.

Formally, Π is **undecidable IFF** there **<u>cannot</u> exist** an algorithm A such that, for every instance I, $A(I)$ returns the correct answer in **finite time**.

Equivalently, Π is **undecidable IFF,** for every algorithm A, there exists some input I such that $A(I)$ **does not** return the correct answer in finite time.

I.e., for some input, $A(I)$ either runs forever or returns the wrong answer

HALTING: AN UNDECIDABLE PROBLEM

Problem 7.19

Halting

Instance: *A computer program A and input x for the program A.*

Question: *When program A is executed with input x, will it halt in finite time?*

For example, you could run $Halt(BFS, G)$ to determine whether, $BFS(G)$ will halt in finite time, which it will, so $Halt(BFS, G)$ returns yes.

The **Halting** problem is **decidable <u>IFF</u>**
there **exists an algorithm $\boldsymbol{Halt(I)}$** that,
for **every** instance $I = (A, x)$,
$Halt(I)$ has **finite** runtime and correctly answers the question:
"would a call to $A(x)$ halt in finite time?"

UNDECIDABILITY OF THE HALTING PROBLEM

Suppose that *Halt* is a program that solves the **Halting Problem**.

We suppose Halt **exists**, to obtain a **contradiction**…

The statement "*Halt* solves the Halting problem" means that *Halt* runs in finite time, and:

$$\mathit{Halt}(A, x) = \begin{cases} \textbf{true} & \text{if } A(x) \text{ halts} \\ \textbf{false} & \text{if } A(x) \text{ doesn't halt.} \end{cases}$$

Note that A (the "algorithm") and x (the "input" to A) are both strings over some finite alphabet.

Since A **is a string** (of code), and its input x **is also a string**…
we ***could*** pass A as an argument to itself: $A(A)$

Then we could ask **if $A(A)$ halts**, by running $Halt(A, A)$…

Weird… Let's try to obtain a contradiction by doing this…

Consider the following algorithm *Strange*.

Algorithm: *Strange*(A)
external *Halt*
if **not** *Halt*(A, A)
  then return (!)
  else $\begin{cases} i \leftarrow 1 \\ \textbf{while } i \neq 0 \textbf{ do } i \leftarrow i + 1 \end{cases}$

If **not** $\boldsymbol{Halt(A, A)}$, then $\boldsymbol{A(A)}$ **will run forever**

but $Strange(A)$ **terminates in finite time**

Else if $\boldsymbol{Halt(A, A)}$, then $\boldsymbol{A(A)}$ **will terminate in finite time**

But $Strange(A)$ **will run forever**

What happens when we run *Strange*(*Strange*)?

Two cases: $Strange(Strange)$ either **halts** or **does not halt**

Suppose $\boldsymbol{Strange(Strange)}$ **halts.** Then, it must return. This means it sees $not\ Halt(A, A)$ just before returning.

But $A = Strange$, so it sees $not\ Halt(Strange, Strange)$.

So, $\boldsymbol{Strange(Strange)}$ **does not halt --- contradiction!**

Suppose $\boldsymbol{Strange(Strange)}$ **does not halt.** Then, it must spin in the while loop forever. This means $Halt(A, A) = true$.

But $A = Strange$, so $Halt(Strange, Strange) = true$.

So, $\boldsymbol{Strange(Strange)}$ **halts --- contradiction!**

Both cases lead to a contradiction. So, our only assumption, **that Halt exists**, must be false!

Therefore, the Halting problem is **undecidable**.

Another Undecidable Problem

Here is another example of an undecidable problem. The problem **Halt-All** takes a program *A* as input and asks if *A* halts on all inputs x.

We describe a Turing reduction **Halting** $\leq^T$ **Halt-All**, which proves that **Halt-All** is undecidable.

Assume we have a program *HaltAllSolver*.

For a fixed program *A* and input x, let $B_x()$ be the program that executes *A*(x) (so B_x has no input).

Here is the reduction:

> Given *A* and x (an instance of **Halting**), construct the program B_x.
> Run *HaltAllSolver*(B_x),

We have

$$\textit{HaltAllSolver}(B_x) = \textbf{true} \quad \Leftrightarrow \quad A(x) \text{ halts},$$

so we can solve the halting problem.

If we have *HaltAllSolver* then we have *Halt*,
but this is impossible, so *HaltAllSolver* cannot exist,
so **the Halt-All problem is undecidable**!

FINISHING NPC

TRANSFORMATIONS/REDUCTIONS

Summary of Polynomial Transformations

Every problem in NP
transforms to

SAT

↓ transforms to

3-SAT

↓

Clique

↓

Vertex Cover

↙

Subset Sum

Summary of Polynomial Transformations

REDUCE **TARGET SUBSET SUM** TO **0-1 KNAPSACK**

RECALL: 0-1 KNAPSACK PROBLEM

Problem 7.3

0-1 Knapsack-Dec

Instance: *a list of* **profits**, $P = [p_1, \ldots, p_n]$; *a list of* **weights**, $W = [w_1, \ldots, w_n]$; *a* **capacity**, M; *and a* **target profit**, T.

Question: *Is there an* n*-tuple* $[x_1, x_2, \ldots, x_n] \in \{0,1\}^n$ *such that* $\sum w_i x_i \leq M$ *and* $\sum p_i x_i \geq T$?

Can I obtain profit T (or better) by taking (whole) items with total weight $\leq M$?

TARGET SUBSET SUM $\leq_P$ 0-1 KNAPSACK

Problem 7.18

Subset Sum

Instance: *A list of* **sizes** $S = [s_1, \ldots, s_n]$; *and a* **target sum**, T. *These are all positive integers.*

Question: *Does there exist a subset* $J \subseteq \{1, \ldots, n\}$ *such that* $\sum_{i \in J} s_i = T$?

Problem 7.3

0-1 Knapsack-Dec

Instance: *a list of* **profits**, $P = [p_1, \ldots, p_n]$; *a list of* **weights**, $W = [w_1, \ldots, w_n]$; *a* **capacity**, M; *and a* **target profit**, T.

Question: *Is there an* n*-tuple* $[x_1, x_2, \ldots, x_n] \in \{0, 1\}^n$ *such that* $\sum w_i x_i \leq M$ *and* $\sum p_i x_i \geq T$?

How should we poly-transform (Target) Subset-Sum input into (Target) 0-1 Knapsack input

Such that: I contains a subset that sums to $\boldsymbol{T}$ **IFF** ($\geq \boldsymbol{T}$) profit can be obtained in knapsack input $f(I)$

Subset Sum $\leq_P$ 0-1 Knapsack

Let I be an instance of **Subset Sum** consisting of **ints** $[s_1, \ldots, s_n]$ and target sum T.

Define

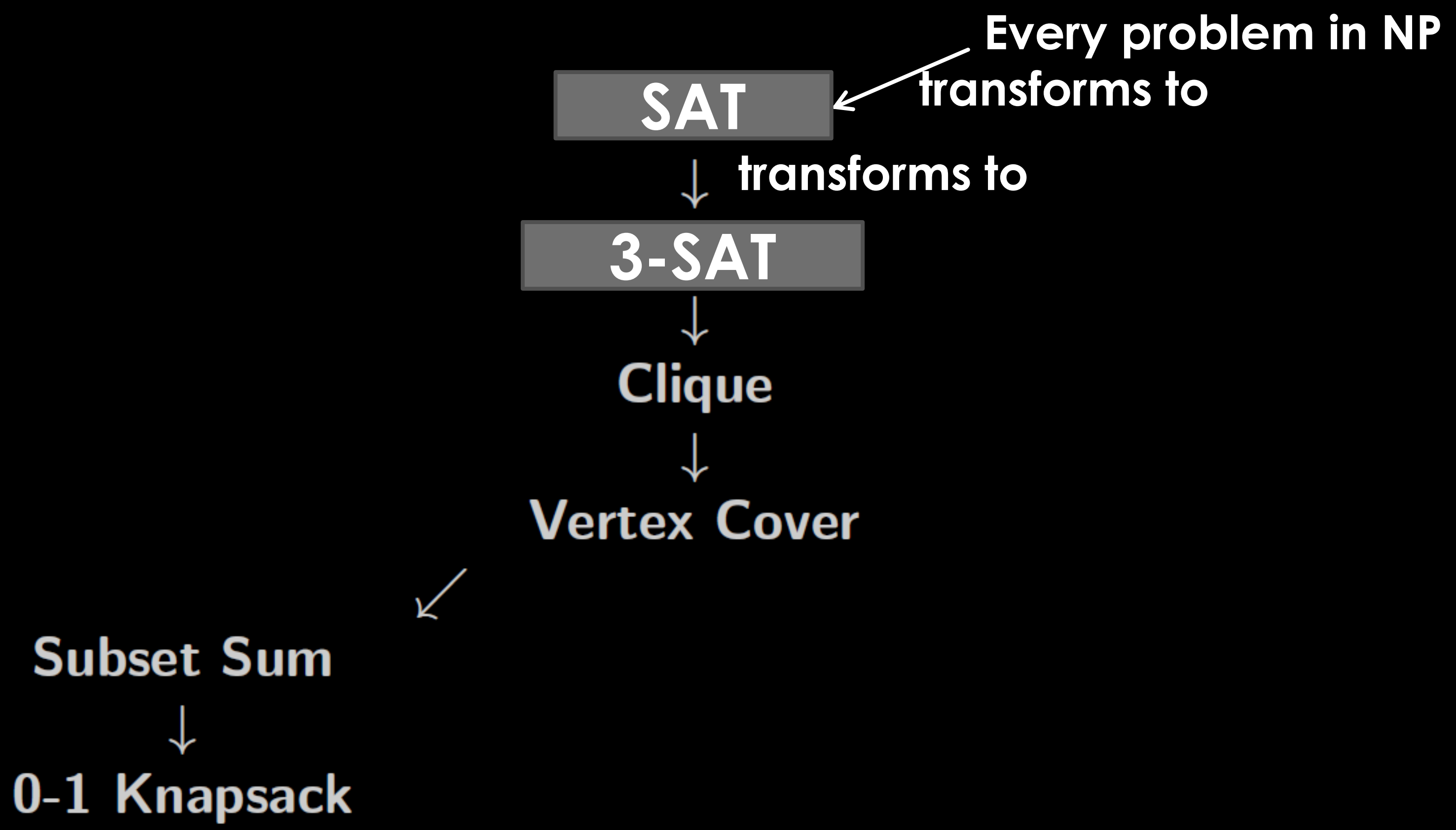
$$p_i = s_i,\ 1 \leq i \leq n$$
$$w_i = s_i,\ 1 \leq i \leq n$$
$$M = T$$

Then define $f(I)$ to be the instance of **0-1 Knapsack** consisting of profits $[p_1, \ldots, p_n]$, weights $[w_1, \ldots, w_n]$, capacity M and target profit T.

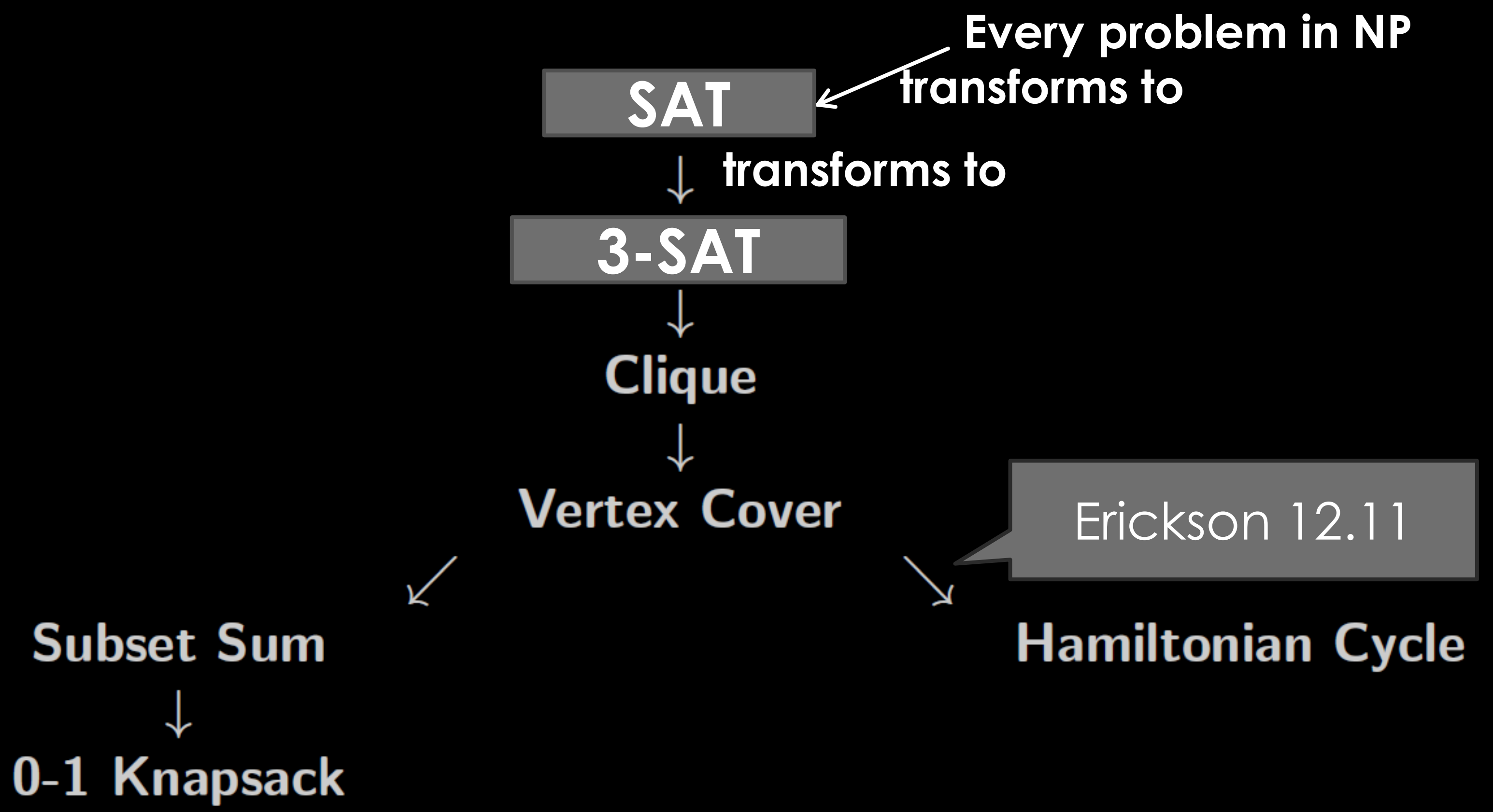
Exercise: Prove the correctness of this transformation.

Claim: I contains a subset that sums to $\boldsymbol{T}$ **IFF** ($\geq \boldsymbol{T}$) profit can be obtained in knapsack input $f(I)$

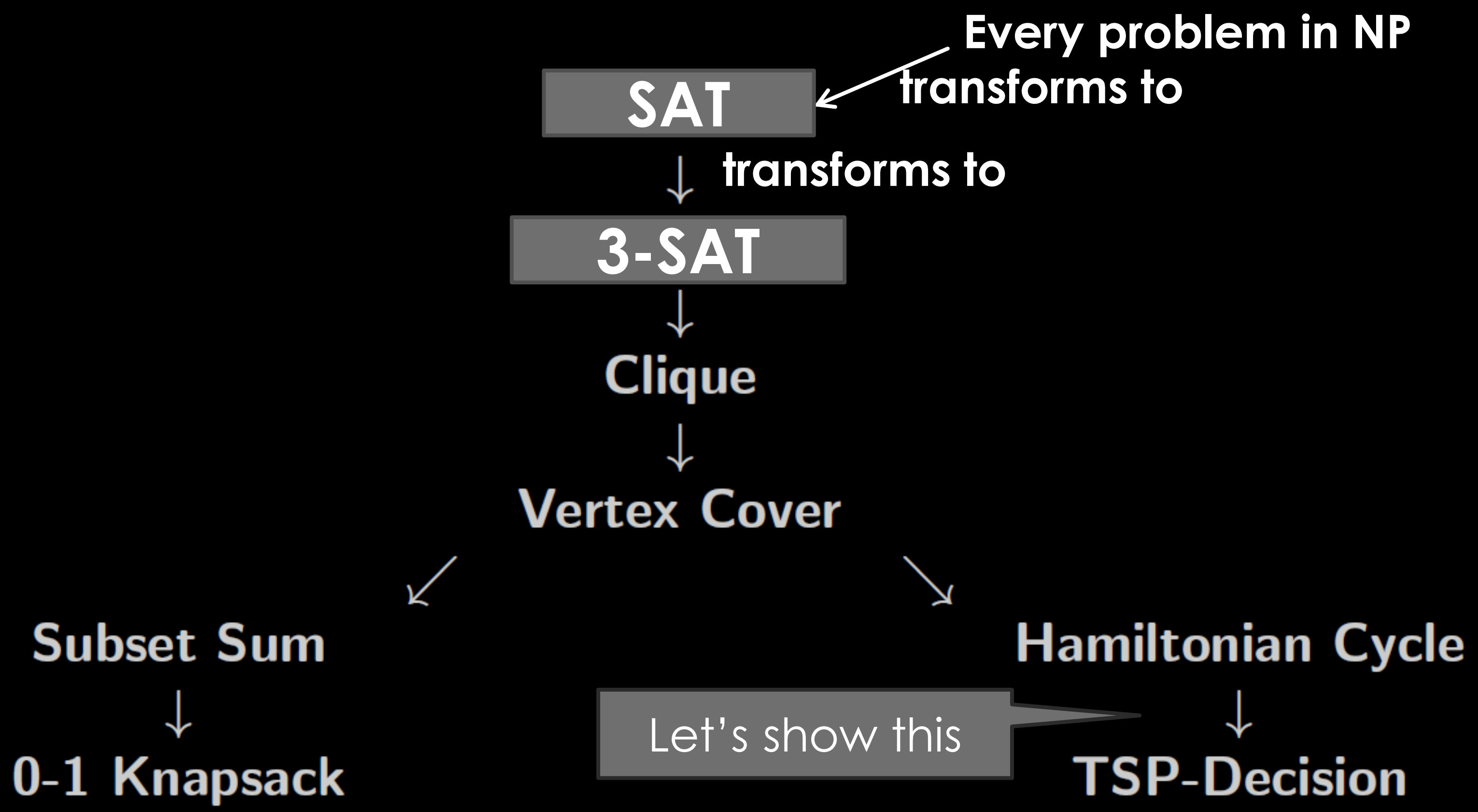
Summary of Polynomial Transformations



Summary of Polynomial Transformations



Summary of Polynomial Transformations



REDUCE **HAMILTONIAN CYCLE** TO **TSP-DECISION**

EXERCISE: GIVE A POLY-TRANSFORMATION

Problem 7.2

Hamiltonian Cycle
Instance: *An undirected graph* $G = (V, E)$.
Question: *Does* G *contain a hamiltonian cycle?*

A **hamiltonian cycle** is a cycle that passes through every vertex in V exactly once.

Problem 7.7

TSP-Decision
Instance: *A graph* G, *edge weights* $w : E \rightarrow \mathbb{Z}^+$, *and a target* T.
Question: *Does there exist a hamiltonian cycle* H *in* G *with* $w(H) \leq T$?

This exercise: Show how to transform Hamiltonian Cycle input into TSP-Decision input (in poly time).

Such that: I contains a Ham Cycle **IFF** $f(I)$ contains a Ham Cycle of weight at most $\boldsymbol{T}$

Hamiltonian Cycle $\leq_P$ TSP-Dec

Let I be an instance of **Hamiltonian Cycle** consisting of a graph $G = (V, E)$.

For the complete graph K_n, where $n = |V|$, define edge weights as follows:

$$w(uv) = \begin{cases} 1 & \text{if } uv \in E \\ 2 & \text{if } uv \notin E. \end{cases}$$

Then define $f(I)$ to be the instance of **TSP-Dec** consisting of the graph K_n, edge weights w and target $T = n$.

Exercise: Prove the correctness of this transformation.

Summary of Polynomial Transformations

Every problem in NP transforms to

SAT

↓ transforms to

3-SAT

↓

Clique

↓

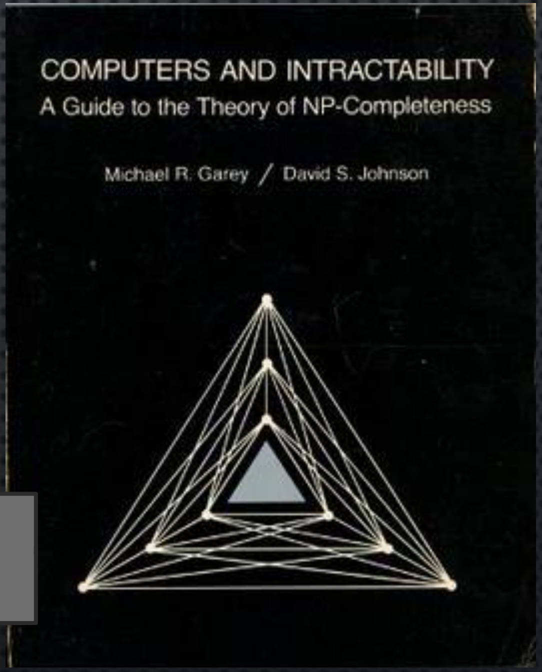
Vertex Cover

↙ ↘

Subset Sum ↓ 0-1 Knapsack

Hamiltonian Cycle ↓ TSP-Decision

Any many, many more ☺…
over 300 listed in this book



FUN AND GAMES

https://arxiv.org/pdf/1203.1895.pdf

Abstract

We prove NP-hardness results for five of Nintendo's largest video game franchises: Mario, Donkey Kong, Legend of Zelda, Metroid, and Pokémon. Our results apply to generalized versions of Super Mario Bros. 1–3, The Lost Levels, and Super Mario World; Donkey Kong Country 1–3; all Legend of Zelda games; all Metroid games; and all Pokémon role-playing games. In addition, we prove PSPACE-completeness of the Donkey Kong Country games and several Legend of Zelda games.

3-SAT $\leq_P^T$ Super Mario

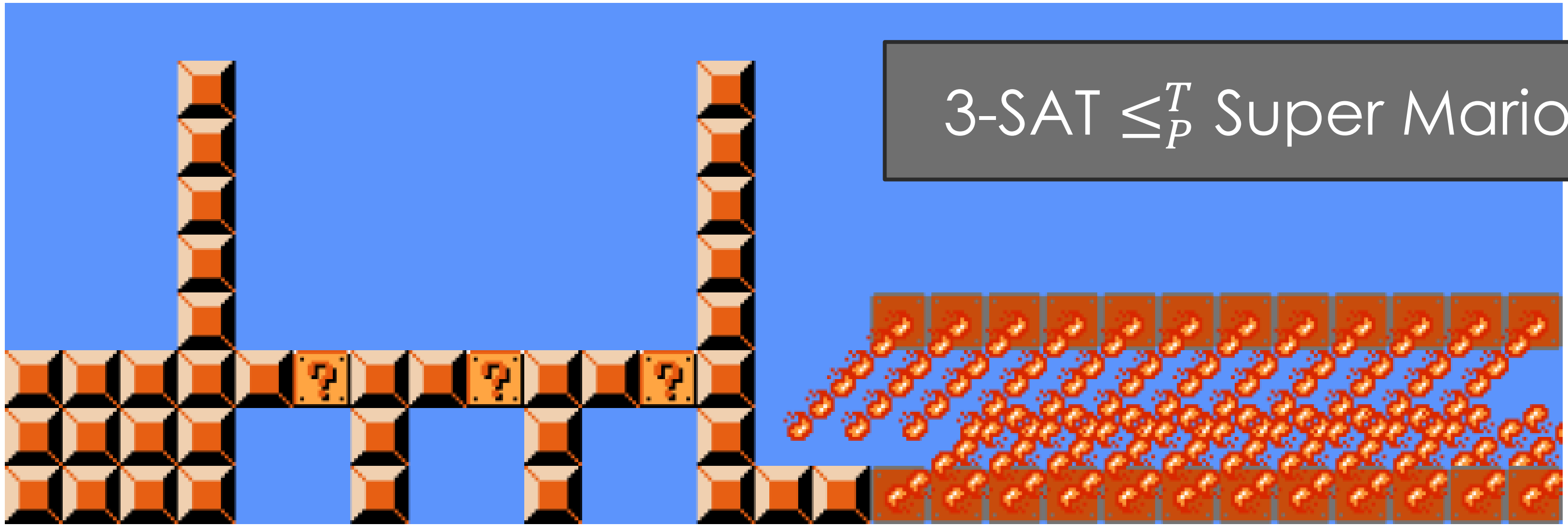


Figure 11: Clause gadget for Super Mario Bros.

(FACTUALLY INCORRECT) MEMES

- There's also an old video meme about proving that Super Mario Bros is NP complete
  - (Long before it was legitimately proved NP hard ☺)
- Whereas the stuff on the **previous slide is real math**, the stuff in this video is just a meme, and is **very wrong.** but you may find it funny…

SUMMARY OF COMPLEXITY CLASSES

See this slide's notes

- **P** (Poly-time)
  E.g., (**decision** problem variants of:) BFS, Dijkstra's, **some** DP algorithms
  - **Decision** problems that can be solved by algorithms with runtime poly(input size)
- **NP** (Non-deterministic poly-time)
  All of P, and e.g.,, vertex cover, clique, SAT, subset sum
  - **Decision** problems for which **certificates** can be **verified** in time poly(input size)
  - Equivalently: decision problems that can be solved in poly-time if you have access to a non-deterministic oracle that returns a yes-certificate if one exists
- **NPC** (NP-complete)
  E.g., vertex cover, clique, SAT, subset sum, TSP-decision
  - **Decision** problems $\Pi \in \boldsymbol{NP}$ s.t. every $\Pi' \in NP$ can be **transformed** to Π in poly-time
- **NP-hard** (at least as hard as NPC)
  All of NPC, and e.g., TSP-optimization, TSP-optimal value
  - problems Π s.t. every $\Pi' \in NP$ can be **reduced** to Π in poly-time

- Note: P, NP and NPC problems are **decidable**

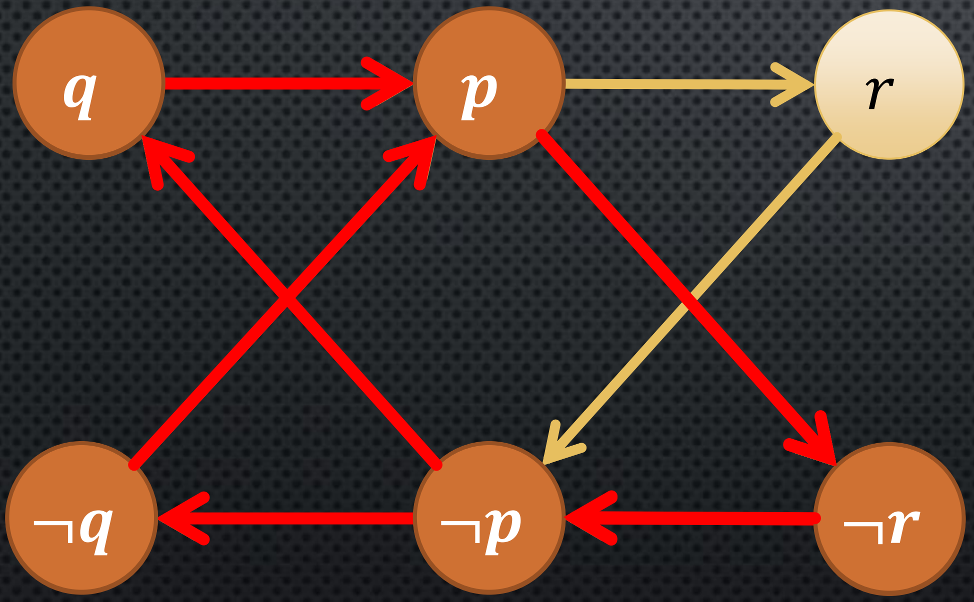
POLYTIME 2-SAT
(IF WE HAVE TIME)

2-SAT EXAMPLES

- $(p \vee q) \wedge (\neg p \vee r) \wedge (\neg r \vee \neg p)$
  - Satisfiable: $p = 0, q = 1, r \in \{0,1\}$
- $(p \vee q) \wedge (\neg p \vee r) \wedge (\neg r \vee \neg p) \wedge (p \vee \neg q)$

Logical refresher:
$p \Rightarrow q$ is **equivalent** to
$\neg p \vee q$.

Therefore, $p \vee q$ is **equivalent** to $\neg p \Rightarrow q$ **and equivalent** to $\neg q \Rightarrow p$



Edges (implications of clauses)…

$\neg p \Rightarrow q$	$p \Rightarrow r$	$r \Rightarrow \neg p$	$\neg p \Rightarrow \neg q$
$\neg q \Rightarrow p$	$\neg r \Rightarrow \neg p$	$p \Rightarrow \neg r$	$q \Rightarrow p$

$q \Rightarrow p \Rightarrow \neg r \Rightarrow \neg p \Rightarrow \neg q$ … so q cannot be $true$

$\neg q \Rightarrow p \Rightarrow \neg r \Rightarrow \neg p \Rightarrow q$ … so q cannot be $false$

Therefore the formula **cannot** be satisfied!

(variable names are integers in 1..|X|)

2-SAT can be solved in polynomial time. Suppose we are given an instance I of **2-SAT** on a set of boolean variables $\boldsymbol{X = \{1..|X|\}}$

(1) For every clause $x \vee y$ (where x and y are literals), construct two directed edges $\overline{x}y$ and $\overline{y}x$. We get a directed graph on vertex set $X \cup \overline{X}$.

(2) Determine the strongly connected components of this directed graph.

(3) I is a yes-instance if and only if there is no strongly connected component containing x and $\overline{x}$, for any $x \in X$.

Suppose no variable x is in the same SCC as $\bar{x}$, then to get a satisfying assignment do the following:
For each x, if ∃ path from x to $\bar{x}$, then set $x = false$ else set $x = true$.