# CS 341: ALGORITHMS

**Lecture 24: intractability VI – Decidability, more NPC transformations**

Readings: see website

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# COMPLEXITY CLASS **EXPTIME**

A very brief overview

**(Non-core material)**

**EXPTIME** is the set of all **decision problems** that can be solved in **exponential time**. I.e., in time $O(2^{poly(n)})$ where $poly(n)$ is a polynomial in the input size.

Observe that $\boldsymbol{NP \subseteq EXPTIME}$

The idea is to generate all possible certificates of an appropriate length and check them for correctness using the given certificate verification algorithm. An an example, for **Hamiltonian Cycle**, we could generate all $n!$ certificates and check each one in turn.

$O(n!) \subseteq O(n^n) = O(2^{n \log n})$ time

We do not know if there are problems in **NP** that cannot be solved in polynomial time (because the **P** $=$ **NP**? conjecture is not yet resolved). However, it is possible to prove that there exist problems in **EXPTIME** $\setminus$ **P**.

One such problem is the **Bounded Halting** problem. Here an instance $I = (A, x, t)$, where $A$ is a program, $x$ is an input to $A$, and $t$ is a positive integer (in binary). The question to be solved is if $A(x)$ halts after at most $t$ computation steps.

The **Bounded Halting** problem can be solved in time $O(t)$, but this is not a polynomial time algorithm because size$(I) = |A| + |x| + \log_2 t$.

Actually, it can be proven that **Bounded Halting** is EXPTIME-complete. This implies that it is in **EXPTIME** $\setminus$ **P**, since it is known that **EXPTIME** $\neq$ **P**.

**$t$ is exponential in $\log t$.**
(And $\log t$ might be the largest term in the input size, in which case $O(t)$ would be **exponential in the input size**.)

# UNDECIDABILITY

Problems that are ***impossible*** to solve

# **DECIDABLE** VS **UNDECIDABLE** PROBLEMS

We say an algorithm $\boldsymbol{A}$ "solves" a decision problem if, for ***every*** instance $I$, $\boldsymbol{A}(\boldsymbol{I})$ has **finite** runtime and returns the correct answer

If an algorithm $A$ **solves** decision problem $\Pi$, then we say $\Pi$ is **decidable**.

Formally, $\Pi$ is **decidable IFF** there exists some algorithm $A$ such that, for **every** instance $I$, $A(I)$ returns the correct answer in **finite** time.

If it is ***not possible*** to design an algorithm $A$ that **solves** decision problem $\Pi$, then we say $\Pi$ is **undecidable**.

Formally, $\Pi$ is **undecidable IFF** there **<u>cannot</u> exist** an algorithm $A$ such that, for every instance $I$, $A(I)$ returns the correct answer in **finite time**.

Equivalently, $\Pi$ is **undecidable IFF,** for every algorithm $A$, there exists some input $I$ such that $A(I)$ **does not** return the correct answer in finite time.

I.e., for some input, $A(I)$ either runs forever or returns the wrong answer

# **HALTING**: AN UNDECIDABLE PROBLEM

**Problem 7.19**

**Halting**

**Instance:** *A computer program A and input $x$ for the program A.*

**Question:** *When program A is executed with input $x$, will it halt in finite time?*

For example, you could run $Halt(BFS, G)$ to determine whether, $BFS(G)$ will halt in finite time, which it will, so $Halt(BFS, G)$ returns yes.

The **Halting** problem is **decidable** **<u>IFF</u>** there exists **an algorithm** $\boldsymbol{Halt(I)}$ that, for **every** instance $I = (A, x)$, $Halt(I)$ has **finite** runtime and correctly answers the question: "would a call to $A(x)$ halt in finite time?"

# **UNDECIDABILITY** OF THE HALTING PROBLEM

Suppose that *Halt* is a program that solves the **Halting Problem**.

We suppose Halt **exists**, to obtain a **contradiction**…

The statement "*Halt* solves the Halting problem" means that *Halt* runs in finite time, and:

$$Halt(A, x) = \begin{cases} \textbf{true} & \text{if } A(x) \text{ halts} \\ \textbf{false} & \text{if } A(x) \text{ doesn't halt.} \end{cases}$$

Note that $A$ (the "algorithm") and $x$ (the "input" to $A$) are both strings over some finite alphabet.

Since $A$ **is a string** (of code), and its input $x$ **is also a string**…
we ***could*** pass $A$ as an argument to itself: $A(A)$

Then we could ask **if $A(A)$ halts**, by running $Halt(A, A)$…

Weird… Let's try to obtain a contradiction by doing this…

Consider the following algorithm *Strange*.

**Algorithm:** *Strange*(*A*)
 **external** *Halt*
 **if** **not** *Halt*(*A*, *A*)
  **then return** (!)
  **else** $\begin{cases} i \leftarrow 1 \\ \textbf{while } i \neq 0 \textbf{ do } i \leftarrow i + 1 \end{cases}$

If $\textbf{not } \boldsymbol{Halt(A,A)}$, then $\boldsymbol{A(A)}$ **will run forever**

but $Strange(A)$ **terminates in finite time**

Else if $\boldsymbol{Halt(A,A)}$, then $\boldsymbol{A(A)}$ **will terminate in finite time**

But $Strange(A)$ **will run forever**

What happens when we run *Strange*(*Strange*)?

Two cases: $Strange(Strange)$ either **halts** or **does not halt**

**Suppose** $\boldsymbol{Strange(Strange)}$ **halts.** Then, it must return.
This means it sees $not\ Halt(A, A)$ just before returning.

But $A = Strange$, so it sees $not\ Halt(Strange, Strange)$.

So, $\boldsymbol{Strange(Strange)}$ **does not halt --- contradiction!**

**Suppose** $\boldsymbol{Strange(Strange)}$ **does not halt.** Then, it must spin in the while loop forever. This means $Halt(A, A) = true$.

But $A = Strange$, so $Halt(Strange, Strange) = true$.

So, $\boldsymbol{Strange(Strange)}$ **halts --- contradiction!**

Both cases lead to a contradiction.
So, our only assumption, **that Halt exists**, must be false!

Therefore, the Halting problem is **undecidable**.

# Another Undecidable Problem

Here is another example of an undecidable problem. The problem **Halt-All** takes a program *A* as input and asks if *A* halts on all inputs $x$.

We describe a Turing reduction **Halting** $\leq^T$ **Halt-All**, which proves that **Halt-All** is undecidable.

Assume we have a program *HaltAllSolver*.

For a fixed program *A* and input $x$, let $B_x()$ be the program that executes *A*$(x)$ (so $B_x$ has no input).

Here is the reduction:

Given *A* and $x$ (an instance of **Halting**), construct the program $B_x$.
Run *HaltAllSolver*$(B_x)$,

We have

$$\textit{HaltAllSolver}(B_x) = \textbf{true} \quad \Leftrightarrow \quad A(x) \text{ halts},$$

so we can solve the halting problem.

If we have *HaltAllSolver* then we have *Halt*,
but this is impossible, so *HaltAllSolver* cannot exist,
so **the Halt-All problem is undecidable**!

# FINISHING NPC

TRANSFORMATIONS/REDUCTIONS

# Summary of Polynomial Transformations

Every problem in NP transforms to → **SAT**

↓ transforms to

**3-SAT**

↓

Clique

↓

Vertex Cover

↙

Subset Sum

# Summary of Polynomial Transformations

Every problem in NP transforms to

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Vertex Cover

↙

(Target) Subset Sum

↓

0-1 Knapsack

Let's show this

# REDUCE **TARGET SUBSET SUM** TO **0-1 KNAPSACK**

# RECALL: 0-1 KNAPSACK PROBLEM

**Problem 7.3**

**0-1 Knapsack-Dec**

**Instance:** *a list of* **profits**, $P = [p_1, \ldots, p_n]$; *a list of* **weights**, $W = [w_1, \ldots, w_n]$; *a* **capacity**, $M$; *and a* **target profit**, $T$.

**Question:** *Is there an* $n$*-tuple* $[x_1, x_2, \ldots, x_n] \in \{0,1\}^n$ *such that* $\sum w_i x_i \leq M$ *and* $\sum p_i x_i \geq T$?

Can I obtain profit $T$ (or better) by taking (whole) items with total weight $\leq M$?

# TARGET SUBSET SUM $\leq_P$ 0-1 KNAPSACK

**Problem 7.18**

**Subset Sum**

**Instance:** *A list of* **sizes** $S = [s_1, \ldots, s_n]$; *and* *a* **target sum**, $T$. *These are all positive integers.*

**Question:** *Does there exist a subset* $J \subseteq \{1, \ldots, n\}$ *such that* $\sum_{i \in J} s_i = T$?

How should we poly-transform (Target) Subset-Sum input into (Target) 0-1 Knapsack input

**Problem 7.3**

**0-1 Knapsack-Dec**

**Instance:** *a list of* **profits**, $P = [p_1, \ldots, p_n]$; *a list of* **weights**, $W = [w_1, \ldots, w_n]$; *a* **capacity**, $M$; *and a* **target profit**, $T$.

**Question:** *Is there an* $n$*-tuple* $[x_1, x_2, \ldots, x_n] \in \{0,1\}^n$ *such that* $\sum w_i x_i \leq M$ *and* $\sum p_i x_i \geq T$?

Such that: $I$ contains a subset that sums to $\boldsymbol{T}$ **IFF** ($\geq \boldsymbol{T}$) profit can be obtained in knapsack input $f(I)$

# Subset Sum $\leq_P$ 0-1 Knapsack

Let $I$ be an instance of **Subset Sum** consisting of **ints** $[s_1, \ldots, s_n]$ and target sum $T$.

Define

$$p_i = s_i,\ 1 \leq i \leq n$$
$$w_i = s_i,\ 1 \leq i \leq n$$
$$M = T$$

Then define $f(I)$ to be the instance of **0-1 Knapsack** consisting of profits $[p_1, \ldots, p_n]$, weights $[w_1, \ldots, w_n]$, capacity $M$ and target profit $T$.

Exercise: Prove the correctness of this transformation.

Claim: $I$ contains a subset that sums to $\boldsymbol{T}$ **IFF** ($\geq \boldsymbol{T}$) profit can be obtained in knapsack input $f(I)$

# Summary of Polynomial Transformations

Every problem in NP
transforms to ↓

**SAT**

↓ transforms to

**3-SAT**

↓

Clique

↓

Vertex Cover

↙

Subset Sum

↓

0-1 Knapsack

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↙ ↘

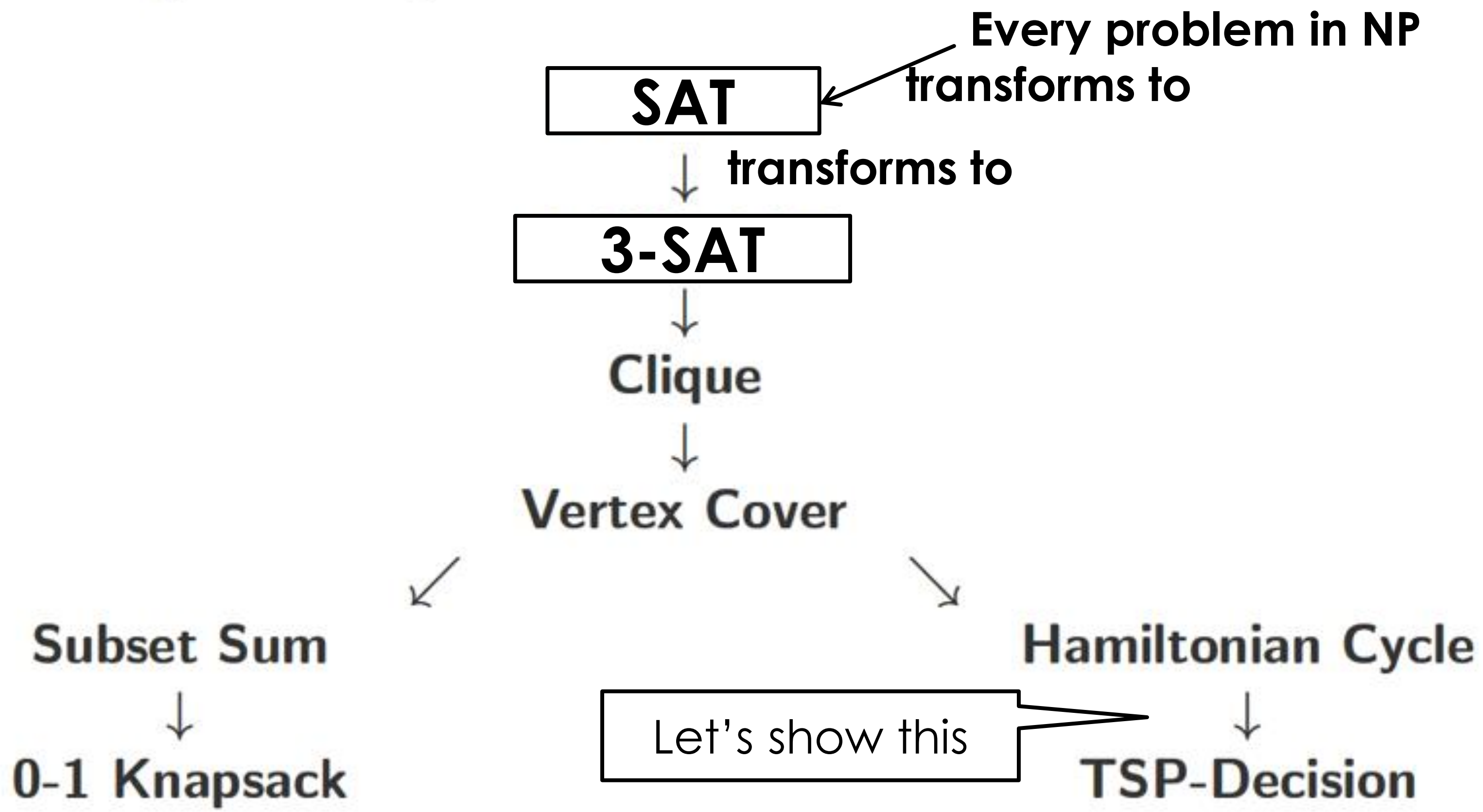
Erickson 12.11

**Subset Sum** | **Hamiltonian Cycle**

↓

**0-1 Knapsack**

# Summary of Polynomial Transformations



# REDUCE **HAMILTONIAN CYCLE** TO **TSP-DECISION**

# **EXERCISE**: GIVE A POLY-TRANSFORMATION

**Problem 7.2**

**Hamiltonian Cycle**
**Instance:** *An undirected graph $G = (V, E)$.*
**Question:** *Does $G$ contain a hamiltonian cycle?*

A **hamiltonian cycle** is a cycle that passes through every vertex in $V$ exactly once.

**Problem 7.7**

**TSP-Decision**
**Instance:** *A graph $G$, edge weights $w : E \to \mathbb{Z}^+$, and a target $T$.*
**Question:** *Does there exist a hamiltonian cycle $H$ in $G$ with $w(H) \leq T$?*

**This exercise:** Show how to transform Hamiltonian Cycle input into TSP-Decision input (in poly time).

**Such that:** $I$ contains a Ham Cycle **IFF** $f(I)$ contains a Ham Cycle of weight at most $\boldsymbol{T}$

# Hamiltonian Cycle $\leq_P$ TSP-Dec

Let $I$ be an instance of **Hamiltonian Cycle** consisting of a graph $G = (V, E)$.

For the complete graph $K_n$, where $n = |V|$, define edge weights as follows:

$$w(uv) = \begin{cases} 1 & \text{if } uv \in E \\ 2 & \text{if } uv \notin E. \end{cases}$$

Then define $f(I)$ to be the instance of **TSP-Dec** consisting of the graph $K_n$, edge weights $w$ and target $T = n$.

Exercise: Prove the correctness of this transformation.

# Summary of Polynomial Transformations

Every problem in NP transforms to

**SAT**

↓ transforms to

**3-SAT**

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↓

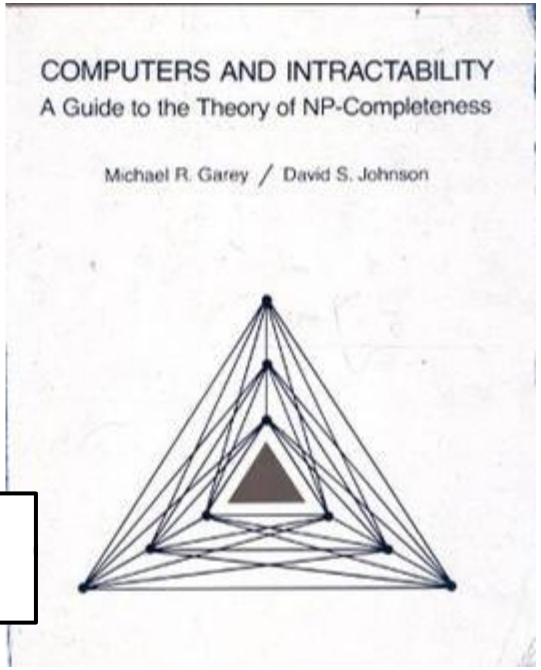
Vertex Cover

↙ ↘

Subset Sum → ↓ 0-1 Knapsack

Hamiltonian Cycle → ↓ TSP-Decision

Any many, many more ☺… over 300 listed in this book

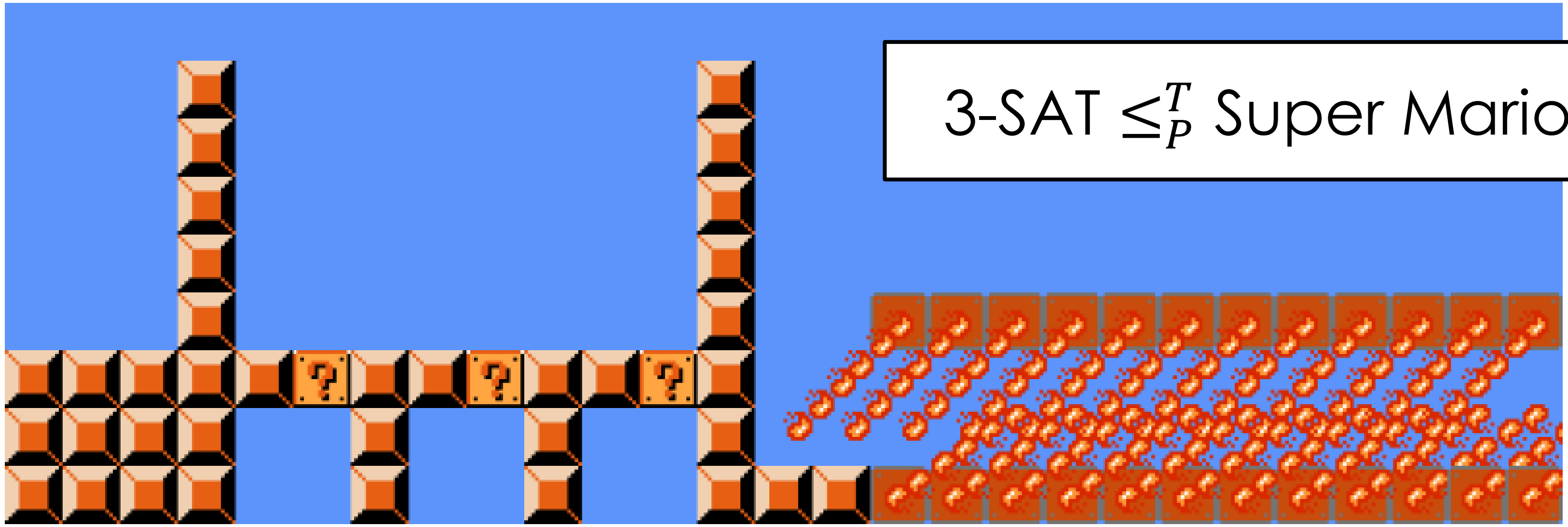


# FUN AND GAMES

https://arxiv.org/pdf/1203.1895.pdf


**Abstract**

We prove NP-hardness results for five of Nintendo's largest video game franchises: Mario, Donkey Kong, Legend of Zelda, Metroid, and Pokémon. Our results apply to generalized versions of Super Mario Bros. 1–3, The Lost Levels, and Super Mario World; Donkey Kong Country 1–3; all Legend of Zelda games; all Metroid games; and all Pokémon role-playing games. In addition, we prove PSPACE-completeness of the Donkey Kong Country games and several Legend of Zelda games.

3-SAT $\leq_P^T$ Super Mario

Figure 11: Clause gadget for Super Mario Bros.

# (FACTUALLY INCORRECT) MEMES

- There's also an old video meme about proving that Super Mario Bros is NP complete
  - (Long before it was legitimately proved NP hard ☺)
- Whereas the stuff on the **previous slide is real math**, the stuff in <u>this video</u> is just a meme, and is **<u>very wrong</u>.** but you may find it funny…

# SUMMARY OF COMPLEXITY CLASSES

See this slide's notes

- **P** (Poly-time)
  - E.g., (**decision** problem variants of:) BFS, Dijkstra's, **some** DP algorithms
  - **Decision** problems that can be solved by algorithms with runtime poly(input size)
- **NP** (Non-deterministic poly-time)
  - **All of P**, and e.g.,, vertex cover, clique, SAT, subset sum
  - **Decision** problems for which **certificates** can be **verified** in time poly(input size)
  - Equivalently: decision problems that can be solved in poly-time if you have access to a non-deterministic oracle that returns a yes-certificate if one exists
- **NPC** (NP-complete)
  - E.g., vertex cover, clique, SAT, subset sum, TSP-decision
  - **Decision** problems $\Pi \in \boldsymbol{NP}$ s.t. every $\Pi' \in NP$ can be **transformed** to $\Pi$ in poly-time
- **NP-hard** (at least as hard as NPC)
  - **All of NPC**, and e.g., TSP-optimization, TSP-optimal value
  - problems $\Pi$ s.t. every $\Pi' \in NP$ can be **reduced** to $\Pi$ in poly-time

- Note: P, NP and NPC problems are **decidable**

# POLYTIME 2-SAT

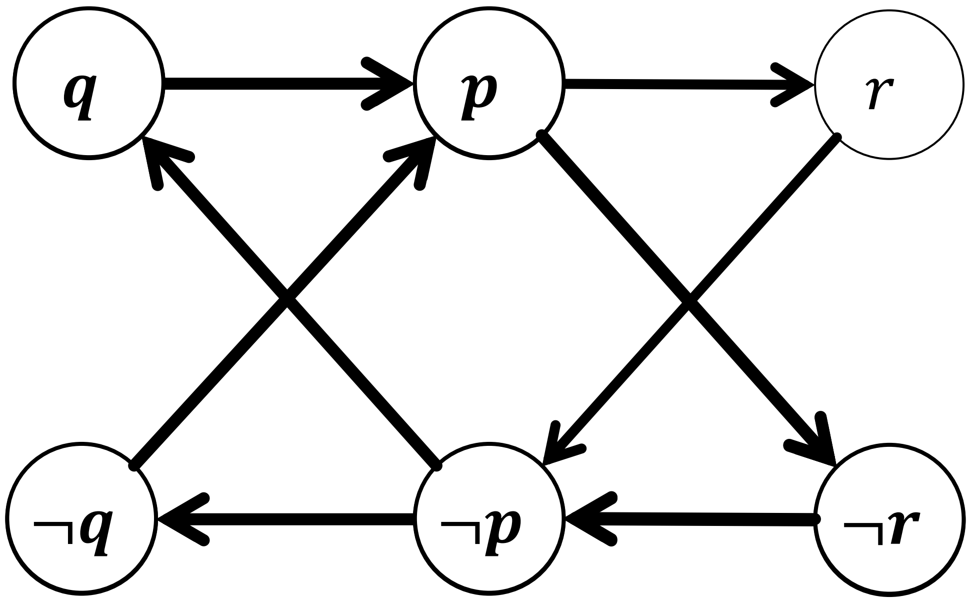
## (IF WE HAVE TIME)

# 2-SAT EXAMPLES

- $(p \lor q) \land (\neg p \lor r) \land (\neg r \lor \neg p)$
  - Satisfiable: $p = 0, q = 1, r \in \{0,1\}$
- $(p \lor q) \land (\neg p \lor r) \land (\neg r \lor \neg p) \land (p \lor \neg q)$

Logical refresher:
$p \Rightarrow q$ is **equivalent** to
$\neg p \lor q$.

Therefore, $p \lor q$ is **equivalent** to $\neg p \Rightarrow q$ **and** **equivalent** to $\neg q \Rightarrow p$




Edges (implications of clauses)…

| | | | |
|---|---|---|---|
| $\neg p \Rightarrow q$ | $p \Rightarrow r$ | $r \Rightarrow \neg p$ | $\neg p \Rightarrow \neg q$ |
| $\neg q \Rightarrow p$ | $\neg r \Rightarrow \neg p$ | $p \Rightarrow \neg r$ | $q \Rightarrow p$ |

$q \Rightarrow p \Rightarrow \neg r \Rightarrow \neg p \Rightarrow \neg q$ … so $q$ cannot be $true$

$\neg q \Rightarrow p \Rightarrow \neg r \Rightarrow \neg p \Rightarrow q$ … so $q$ cannot be $false$

Therefore the formula **cannot** be satisfied!

(variable names are integers in 1..|X|)

**2-SAT** can be solved in polynomial time. Suppose we are given an instance $I$ of **2-SAT** on a set of boolean variables $\boldsymbol{X = \{1..|X|\}}$

(1) For every clause $x \vee y$ (where $x$ and $y$ are literals), construct two directed edges $\overline{x}y$ and $\overline{y}x$. We get a directed graph on vertex set $X \cup \overline{X}$.

(2) Determine the strongly connected components of this directed graph.

(3) $I$ is a yes-instance if and only if there is no strongly connected component containing $x$ and $\overline{x}$, for any $x \in X$.

Suppose no variable $x$ is in the same SCC as $\bar{x}$, then to get a satisfying assignment do the following:
For each $x$, if ∃ path from $x$ to $\bar{x}$, then set $x = false$ else set $x = true$.