CS 341: ALGORITHMS

Lecture 4: divide & conquer III

Readings: see website

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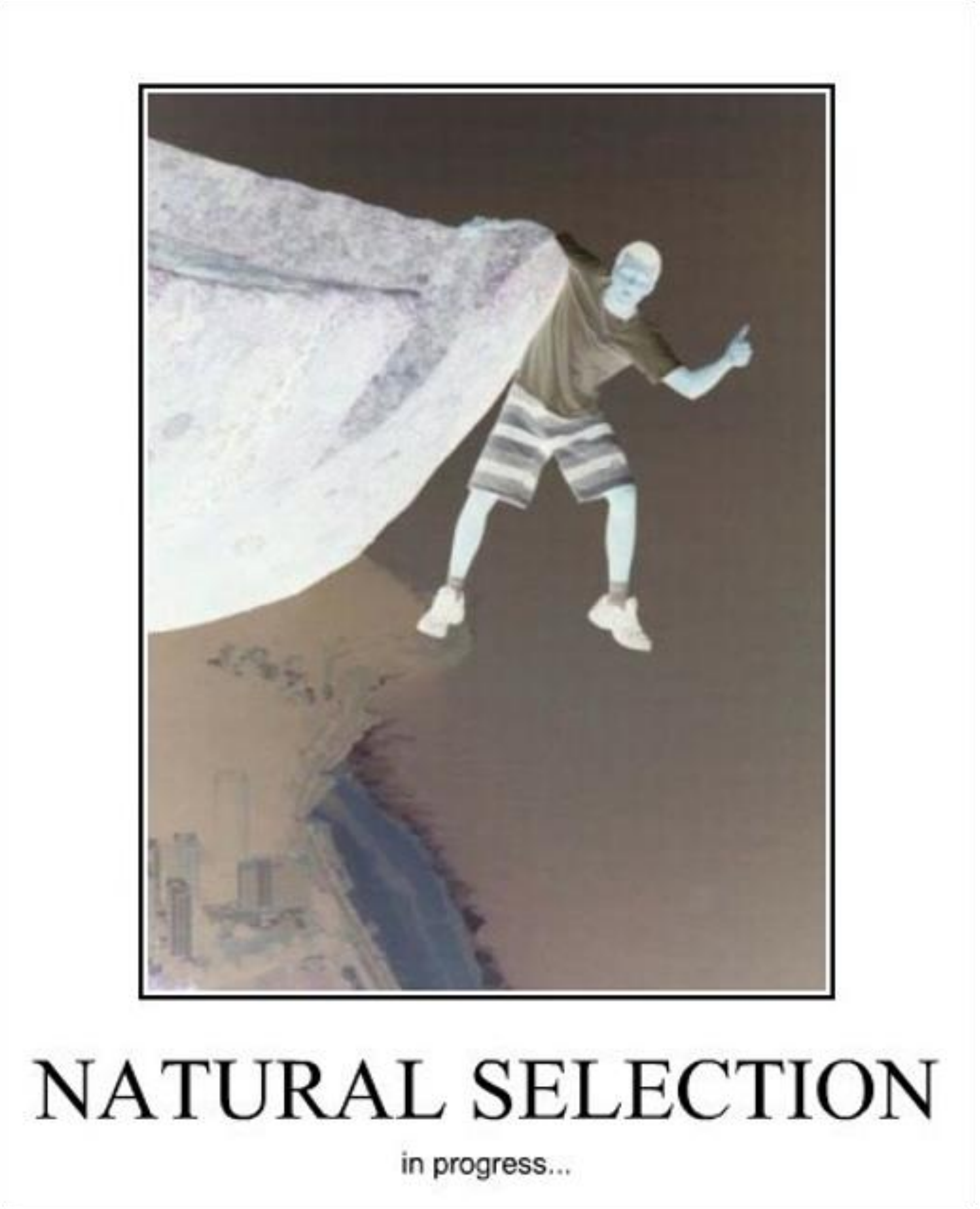
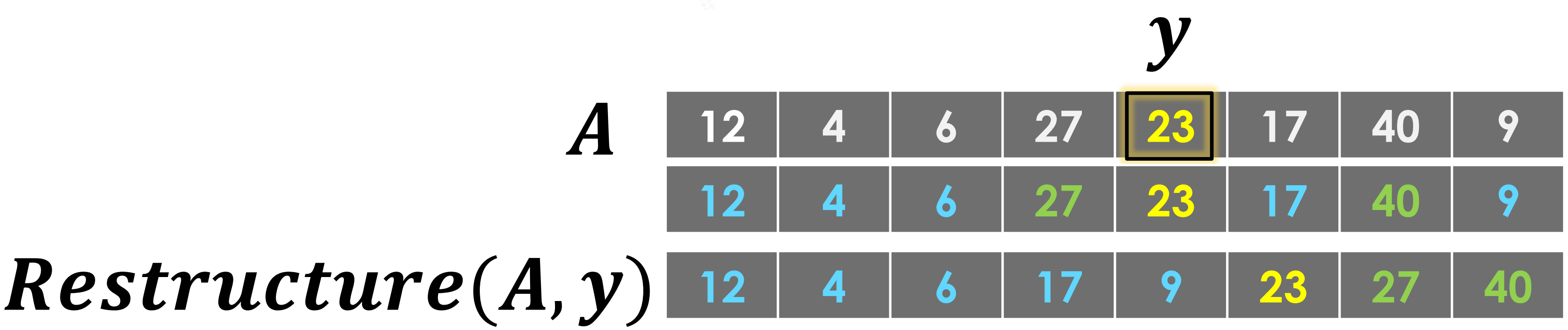
THE **SELECTION** PROBLEM



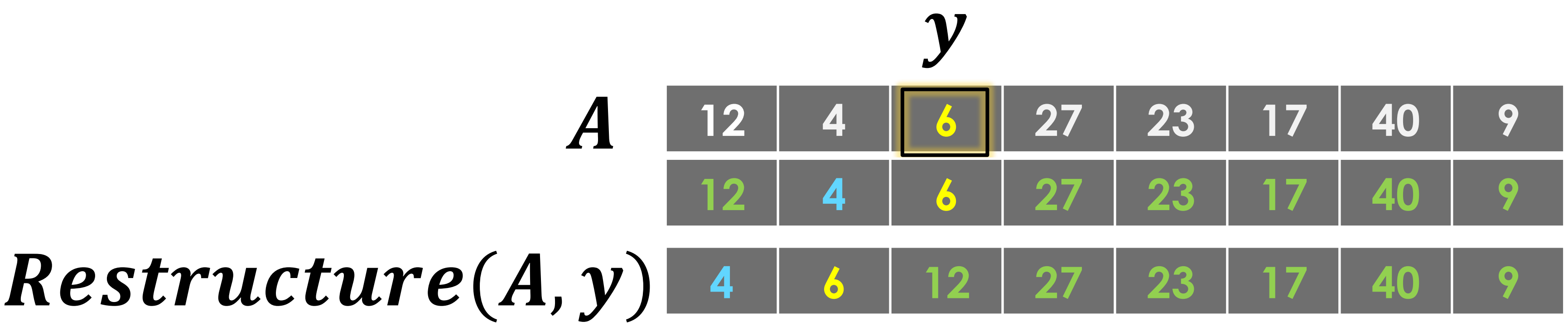
THE **SELECTION** PROBLEM

- Input: An array **A** containing **n** **distinct** integer values, and an integer **k** between 1 and **n**
- Output: The **k**-th **smallest** integer in **A**
- **Minimum** is a special case where $k = 1$
- **Median** is a special case where $k = \frac{n}{2}$
- **Maximum** is a special case where $k = n$
- Simple algorithm for solving selection?

Suppose we choose a **pivot** element y in the array A, and we **restructure** A so that all elements less than y precede y in A, and all elements greater than y occur after y in A. (This is exactly what is done in *Quicksort*, and it takes **linear time**.)



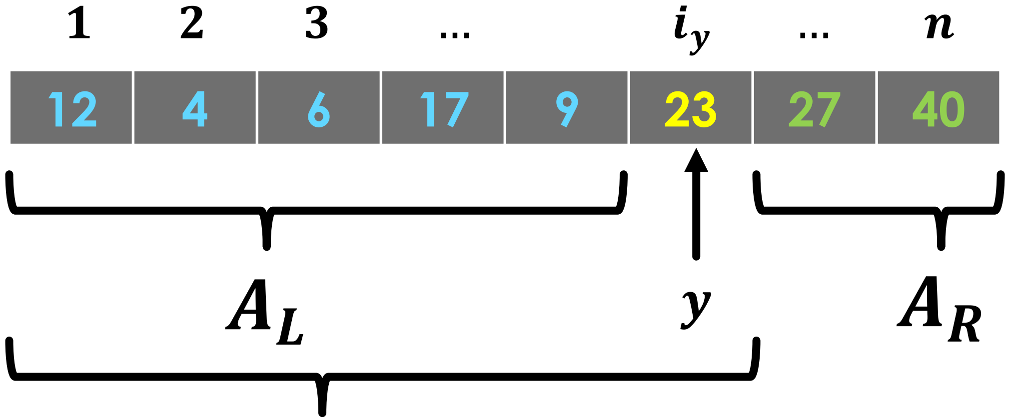
					y			
A	12	4	6	27	23	17	40	9
	12	4	6	27	23	17	40	9
$Restructure(A, y)$	12	4	6	17	9	23	27	40



			y					
A	12	4	6	27	23	17	40	9
	12	4	6	27	23	17	40	9
$Restructure(A, y)$	4	6	12	27	23	17	40	9

Number of elements on each side depend on the **value** y…

$\boldsymbol{A}$ after $\boldsymbol{Restructure(A, y)}$

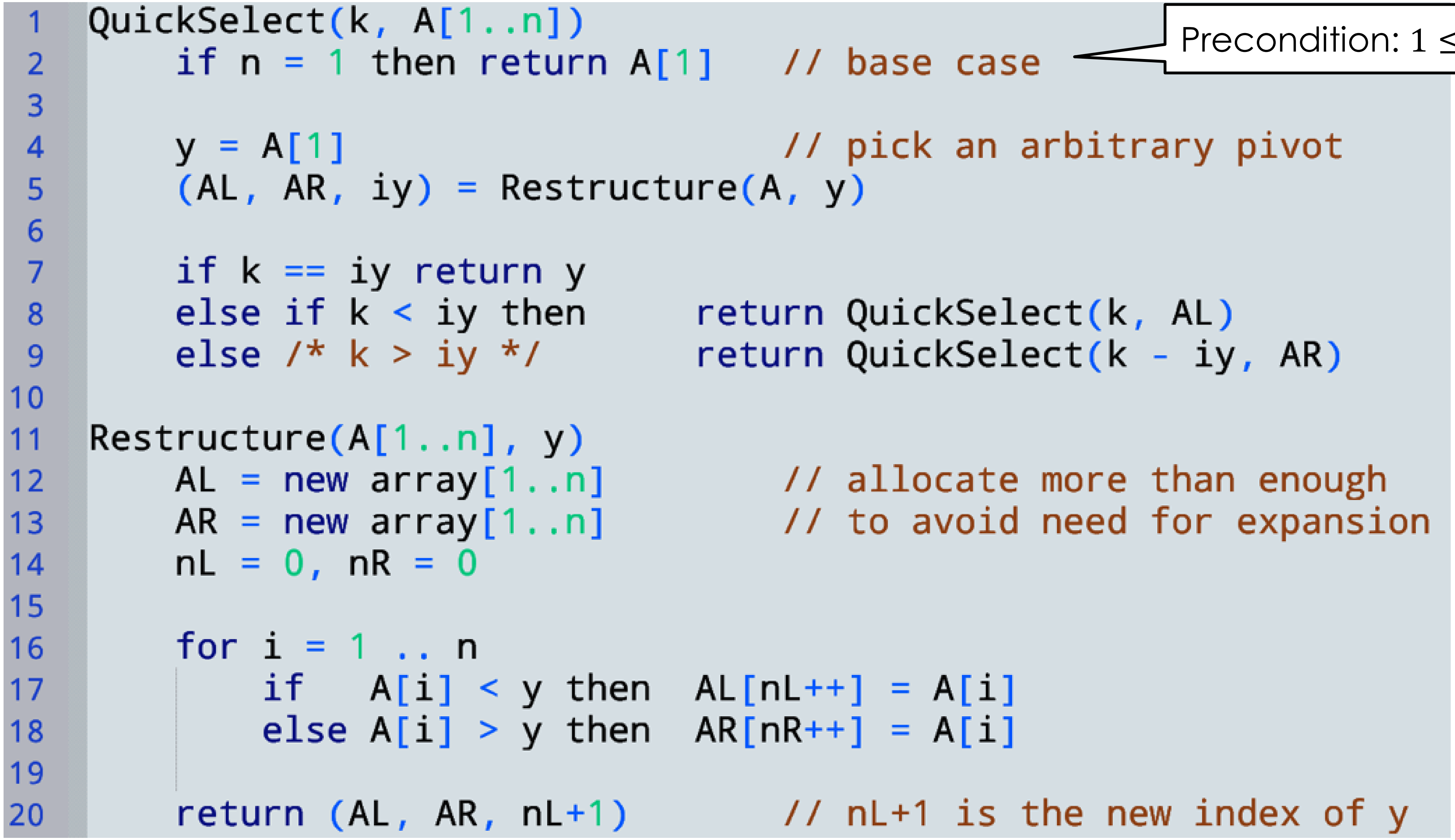


Number of elements in this range = i_y

- **What's the k-th smallest element of A?**
  - If $\boldsymbol{k = i_y}$ then y
  - If $\boldsymbol{k < i_y}$ then the kth smallest in A_L
  - If $\boldsymbol{k > i_y}$ then the $(k - \boldsymbol{i_y})$th smallest in A_R

Recursive calls

Precondition: $1 \leq k \leq n$



```
QuickSelect(k, A[1..n])
    if n = 1 then return A[1]    // base case

    y = A[1]                     // pick an arbitrary pivot
    (AL, AR, iy) = Restructure(A, y)

    if k == iy return y
    else if k < iy then      return QuickSelect(k, AL)
    else /* k > iy */        return QuickSelect(k - iy, AR)

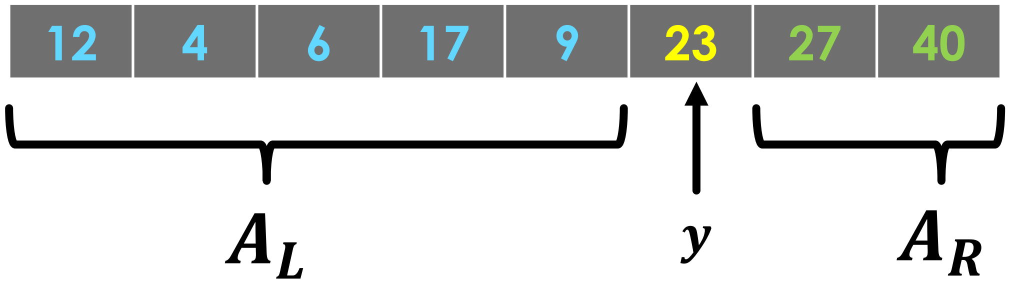
Restructure(A[1..n], y)
    AL = new array[1..n]         // allocate more than enough
    AR = new array[1..n]         // to avoid need for expansion
    nL = 0, nR = 0

    for i = 1 .. n
        if   A[i] < y then   AL[nL++] = A[i]
        else A[i] > y then   AR[nR++] = A[i]

    return (AL, AR, nL+1)        // nL+1 is the new index of y
```

OVERLY OPTIMISTIC ANALYSIS ☺

$\boldsymbol{A}$ after $\boldsymbol{Restructure(A, y)}$



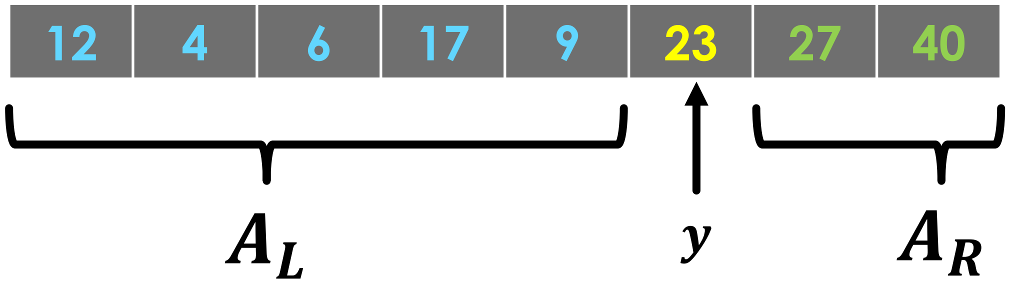
12	4	6	17	9	23	27	40

- If $i_y = \frac{n}{2}$, then we recurse on $\sim\frac{n}{2}$ elements,
- If we could **always** recurse on $\frac{n}{2}$ elements then
    - We would get $T(n) = T\left(\frac{n}{2}\right) + \Theta(n)$
    - Which would yield $a = 1, b = 2, y = 1, x = \log_2 1 = 0$,
      $y > x$ and $T(n) \in \Theta(n^y) = \boldsymbol{\Theta(n)}$ by the Master theorem.

But we **don't** always recurse on $\frac{n}{2}$ elements!

WORST-CASE ANALYSIS

$\boldsymbol{A}$ after $\boldsymbol{Restructure(A, y)}$

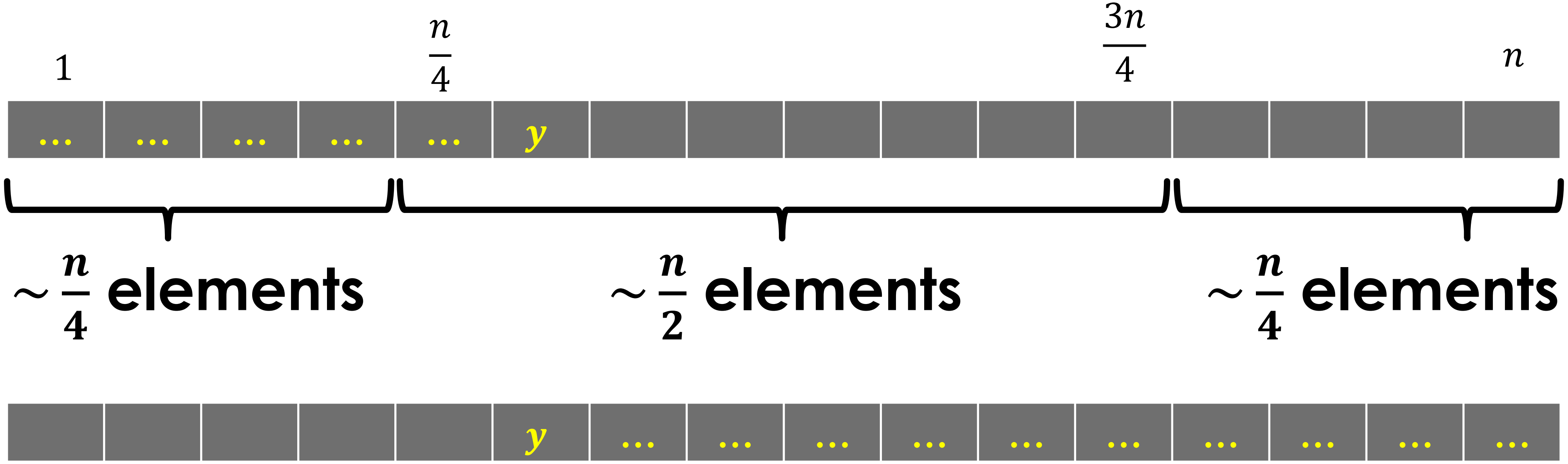


- If we always get $i_y = 1$ and recurse on the right, then
- We get $T(n) = T(n-1) + \Theta(n)$
- By the substitution method this is $\boldsymbol{\Theta(n^2)}$

- So, sometimes the pivot is good, sometimes it's bad…
- **What about the average case?**

AVERAGE-CASE ANALYSIS

- Definition: we say a pivot y is **good** if $\boldsymbol{i_y} \in \left(\frac{\boldsymbol{n}}{\boldsymbol{4}}, \frac{\boldsymbol{3n}}{\boldsymbol{4}}\right)$



- For **any good pivot** we recurse on at most $\frac{3n}{4}$ elements

Reducing the size of the subproblem by at least 1/4

- Probability of an arbitrary pivot being **good**?

p = 1/2

PROOF SKETCH

- Since probability of a good pivot is ½,
- on average, every **two** recursive calls, we will encounter a **good pivot**
- Encountering a good pivot reduces problem size to at most $\frac{3n}{4}$
- So, problem size is reduced to $\frac{3n}{4}$ after **expected linear work**
- **Average case recurrence:** $\boldsymbol{T(n) = T\left(\frac{3n}{4}\right) + \Theta(n)}$
  - $\boldsymbol{T(n) \in \Theta(n)}$

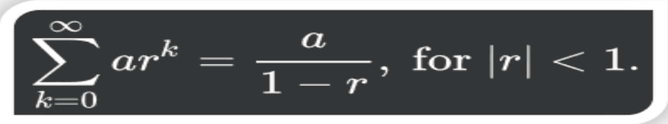
Here is a more rigorous proof of the average-case complexity: We say the algorithm is in phase j if the current subarray has size s, where

$$n\left(\frac{3}{4}\right)^{j+1} < s \leq n\left(\frac{3}{4}\right)^{j}.$$

This is just for your notes, in case you want to know how you'd do this analysis formally

Let X_j be a random variable that denotes the amount of computation time occurring in phase j. If the pivot is in the middle half of the current subarray, then we transition from phase j to phase $j+1$. This occurs with probability $1/2$, so the expected number of recursive calls in phase j is 2. The computing time for each recursive call is linear in the size of the current subarray, so $E[X_j] \leq 2cn(3/4)^j$ (where $E[\cdot]$ denotes the expectation of a random variable). The total time of the algorithm is given by $X = \sum_{j\geq 0} X_j$. Therefore

$$E[X] = \sum_{j\geq 0} E[X_j] \leq 2cn \sum_{j\geq 0} (3/4)^j = 8cn \in O(n).$$



$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}, \text{ for } |r| < 1.$$

TAKING **SELECTION** FURTHER

- We just showed:
  - QuickSelect with **average case** runtime in $O(n)$
- Next up:
  - Median-of-medians QuickSelect (MOMQuickSelect)
  - **worst case** runtime in $O(n)$

Relies on getting a **good pivot** within $O(1)$ recursive calls **on average**

Must get a **good pivot** within $O(1)$ recursive calls **always**

The algorithm we will see picks a **good pivot** in **every** recursive call

HIGH LEVEL ALGORITHM

- Similar to QuickSelect
    - **Choose** a pivot
    - Move smaller elements to the left of the pivot, and larger elements to the right of the pivot
    - Recursively call MOMQuickSelect on one subarray
- Only difference is **how** we choose the pivot
    - **<u>Always</u>** want to pick a **good pivot**

ALWAYS PICKING A **GOOD PIVOT**

Example input A[1...50]:

11, 38, 6, 21, 20, 17, 14, 9, 7, 5, 8, 34, 49, 47, 28, 18, 44, 31, 46, 48, 27, 4, 2, 50, 23, 45, 3, 13, 43, 22, 10, 32, 35, 41, 24, 1, 30, 12, 15, 26, 16, 19, 36, 33, 37, 39, 25, 40, 29, 42

Group into **rows of 5**

11	38	6	21	20
17	14	9	7	5
8	34	49	47	28
18	44	31	46	48
27	4	2	50	23
45	3	13	43	22
10	32	35	41	24
1	30	12	15	26
16	19	36	33	37
39	25	40	29	42

Time complexity for this step?

Find **median** of **each row**

11	38	6	21	**20**
17	14	**9**	7	5
8	**34**	49	47	28
18	**44**	31	46	48
27	4	2	50	**23**
45	3	13	43	**22**
10	**32**	35	41	24
1	30	12	**15**	26
16	19	36	**33**	37
39	25	40	29	42

Time complexity for this step?

Build **array of medians**

20, 9, 34, 44, 23, 22, 32, 15, 33, 39

Time complexity?

Recursively find the median of these medians: **23**

Recursive problem size?

HOW GOOD IS THE PIVOT **<u>23</u>**?

elements ≤ 23 is **at least** $3(5)$.
This is at least 3/10ths of our 50-element input, or $\mathbf{3n/10}$.

Recall: median of each row

11	38	6	21	**20**
17	14	**9**	7	5
8	**34**	49	47	28
18	**44**	31	46	48
27	4	2	50	***<u>23</u>***
45	3	13	43	**22**
10	**32**	35	41	24
1	30	12	**15**	26
16	19	36	**33**	37
39	25	40	29	42

<u>Imagine</u> sorting each row:

6	11	**20**
5	7	**9**
8	28	**34**
18	31	**44**
2	4	***<u>23</u>***
3	13	**22**
10	24	**32**
1	12	**15**
16	19	**33**
25	29	**39**

Then ordering r by medians:

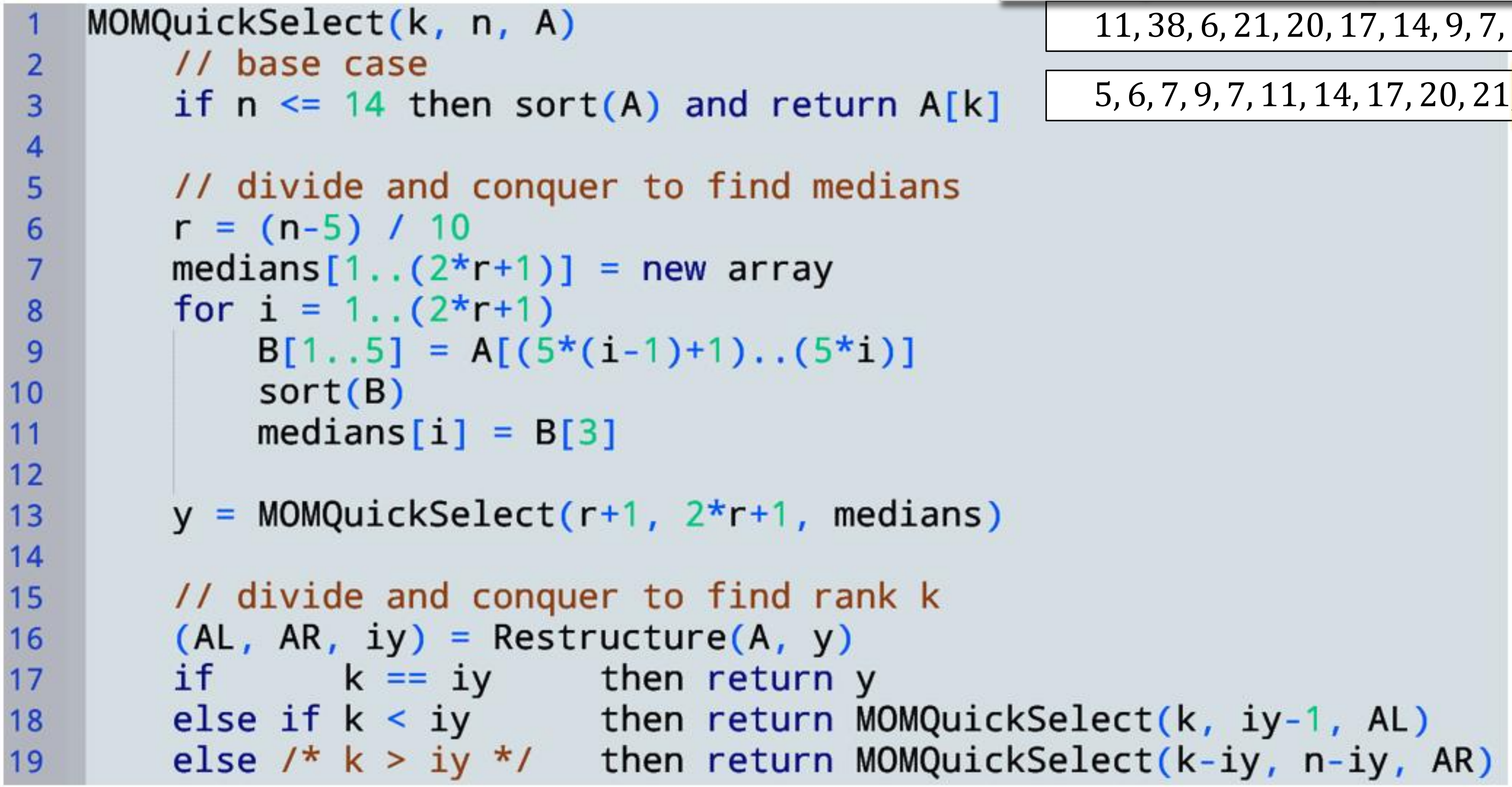
5	7	**9**	14	17
1	12	**15**	26	30
6	11	**20**	21	38
3	13	**22**	43	45
2	4	***<u>23</u>***	27	50
10	24	**32**	35	41
16	19	**33**	36	37
8	28	**34**	47	49
25	29	**39**	40	42
18	31	**44**	46	48

elements ≥ 23 is **at least** $3(6)$.
This is > **3/10ths** of our 50-element input.

So, after restructuring, pivot 23 must have **at least** $\mathbf{3n/10}$ elements **before *<u>and</u>* after** it

This is a **good** pivot!

We recurse on A_L or A_R, and both have size **at most** $\mathbf{7n/10}$



```
MOMQuickSelect(k, n, A)
    // base case
    if n <= 14 then sort(A) and return A[k]

    // divide and conquer to find medians
    r = (n-5) / 10
    medians[1..(2*r+1)] = new array
    for i = 1..(2*r+1)
        B[1..5] = A[(5*(i-1)+1)..(5*i)]
        sort(B)
        medians[i] = B[3]

    y = MOMQuickSelect(r+1, 2*r+1, medians)

    // divide and conquer to find rank k
    (AL, AR, iy) = Restructure(A, y)
    if      k == iy      then return y
    else if k < iy       then return MOMQuickSelect(k, iy-1, AL)
    else /* k > iy */    then return MOMQuickSelect(k-iy, n-iy, AR)
```

$\boldsymbol{MOMQuickSelect(k = 11, n = 14, A)}$

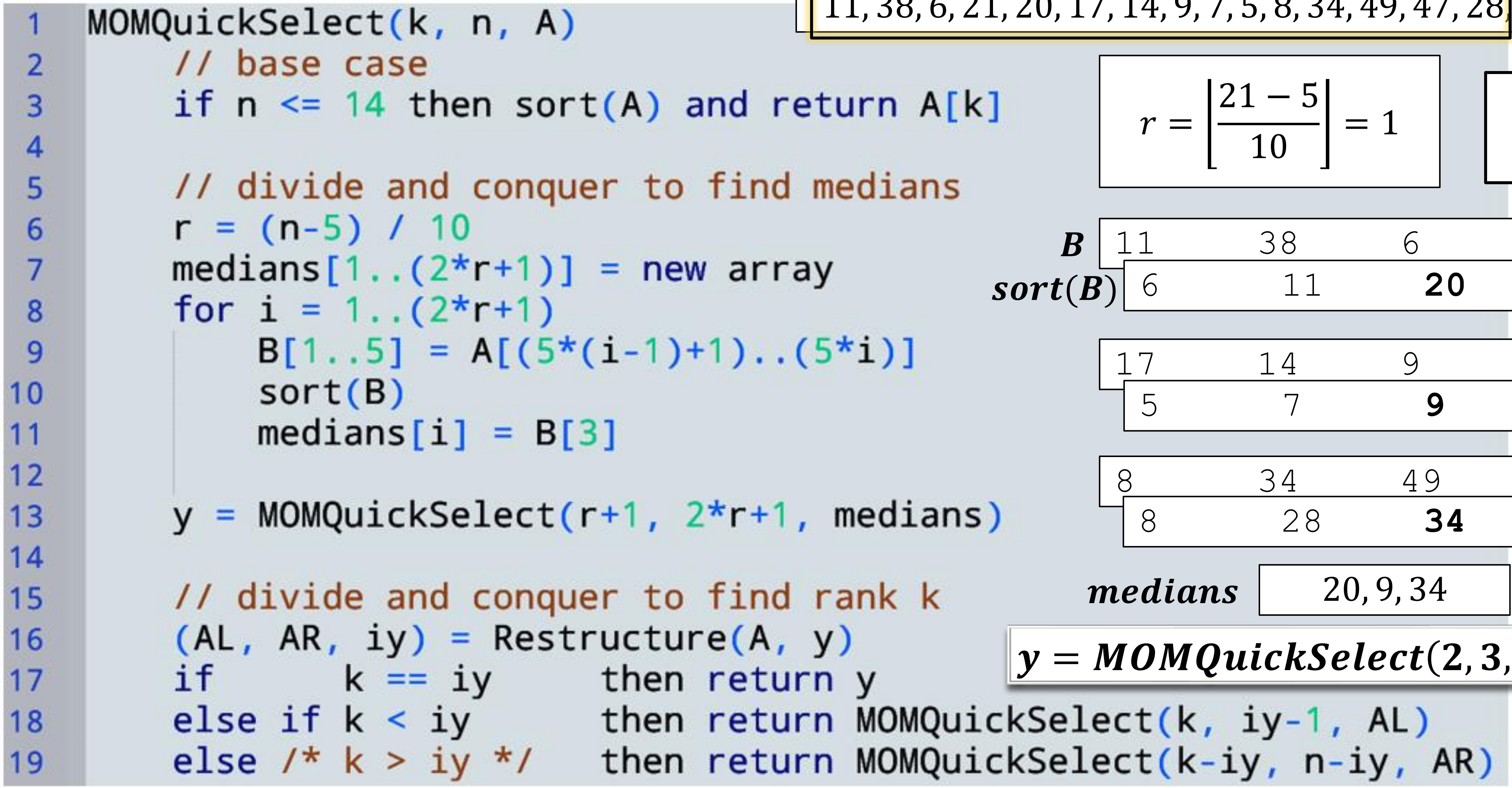
$11, 38, 6, 21, 20, 17, 14, 9, 7, 5, 8, 34, 49, 47$

$5, 6, 7, 9, 7, 11, 14, 17, 20, 21, \boxed{34,} 38, 47, 49$

$\boldsymbol{MOMQuickSelect(k=11, n=21, A)}$

$11, 38, 6, 21, 20, 17, 14, 9, 7, 5, 8, 34, 49, 47, 28, 18, 44, 31, 46, 48, 27$

Not considering at most 9 elements



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    if      k == iy     then return y
    else if k < iy      then return MOMQuickSelect(k, iy-1, AL)
    else /* k > iy */   then return MOMQuickSelect(k-iy, n-iy, AR)
```

$$r = \left\lfloor \frac{21-5}{10} \right\rfloor = 1$$

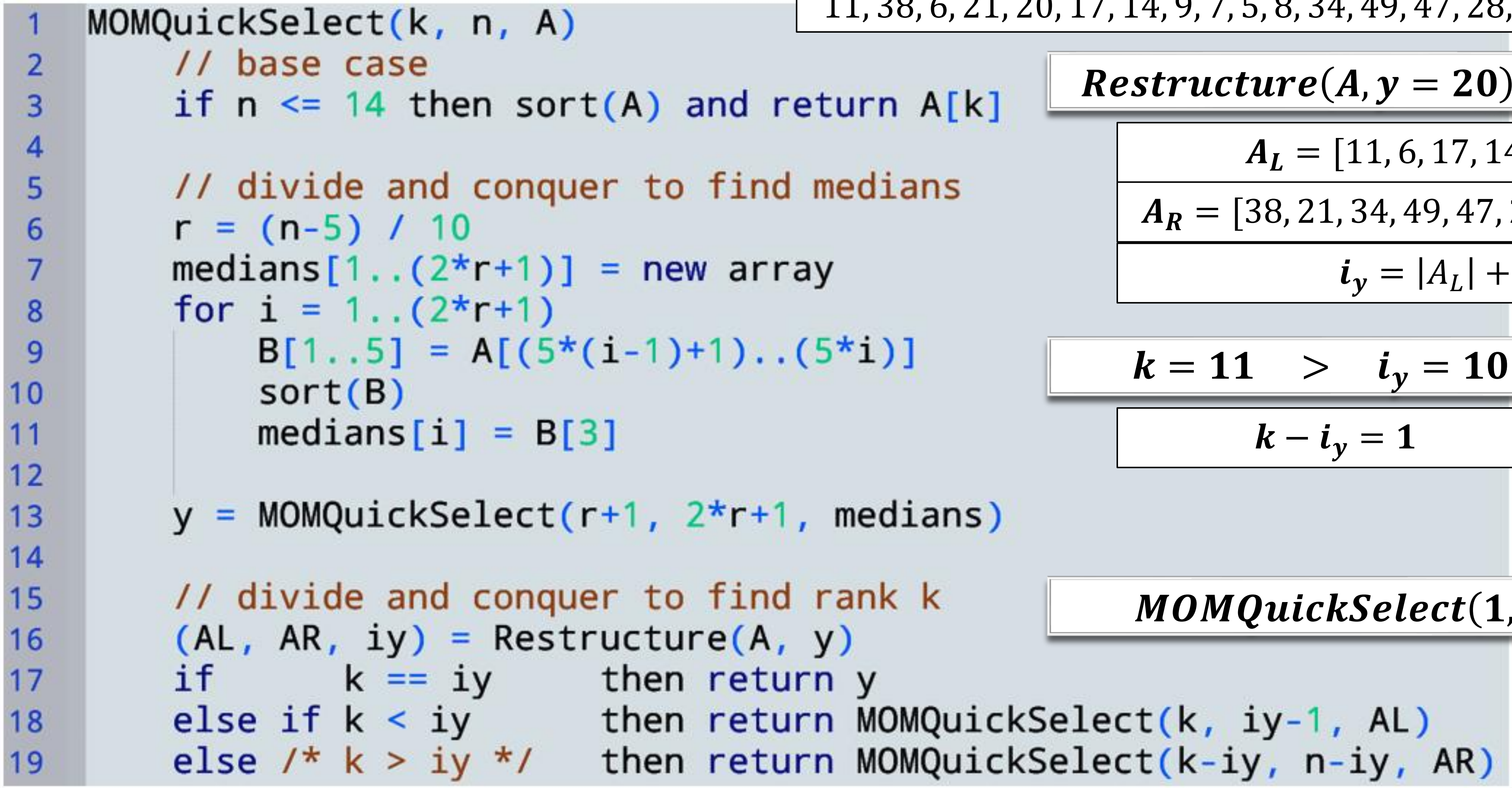
$\boldsymbol{B}$	11	38	6	21	20
$\boldsymbol{sort(B)}$	6	11	**20**	21	38
	17	14	9	7	5
	5	7	**9**	14	17
	8	34	49	47	28
	8	28	**34**	47	49

$\boldsymbol{medians}$ $20, 9, 34$

$\boldsymbol{y = MOMQuickSelect(2, 3, [20, 9, 34]) \Rightarrow 20}$

$MOMQuickSelect(k = 11, n = 21, A)$

$11, 38, 6, 21, 20, 17, 14, 9, 7, 5, 8, 34, 49, 47, 28, 18, 44, 31, 46, 48, 27$



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    else /* k > iy */  then return MOMQuickSelect(k-iy, n-iy, AR)
```

$Restructure(A, y = 20) \Rightarrow$

$A_L = [11, 6, 17, 14, 9, 7, 5, 8, 18]$

$A_R = [38, 21, 34, 49, 47, 28, 44, 31, 46, 48, 27]$

$i_y = |A_L| + 1 = 10$

$k = 11 \quad > \quad i_y = 10$

$k - i_y = 1 \qquad n - i_y = 10$

$MOMQuickSelect(1, 10, A_R) \Rightarrow 21$

Runtime? (unit cost)

```
// base case
if n <= 14 then sort(A) and return A[k]

// divide and conquer to find medians
r = (n-5) / 10
medians[1..(2*r+1)] = new array
for i = 1..(2*r+1)
    B[1..5] = A[(5*(i-1)+1)..(5*i)]
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```

$\Theta(1)$

$\Theta(n)$

$\Theta(n)$ iterations

$\Theta(1)$

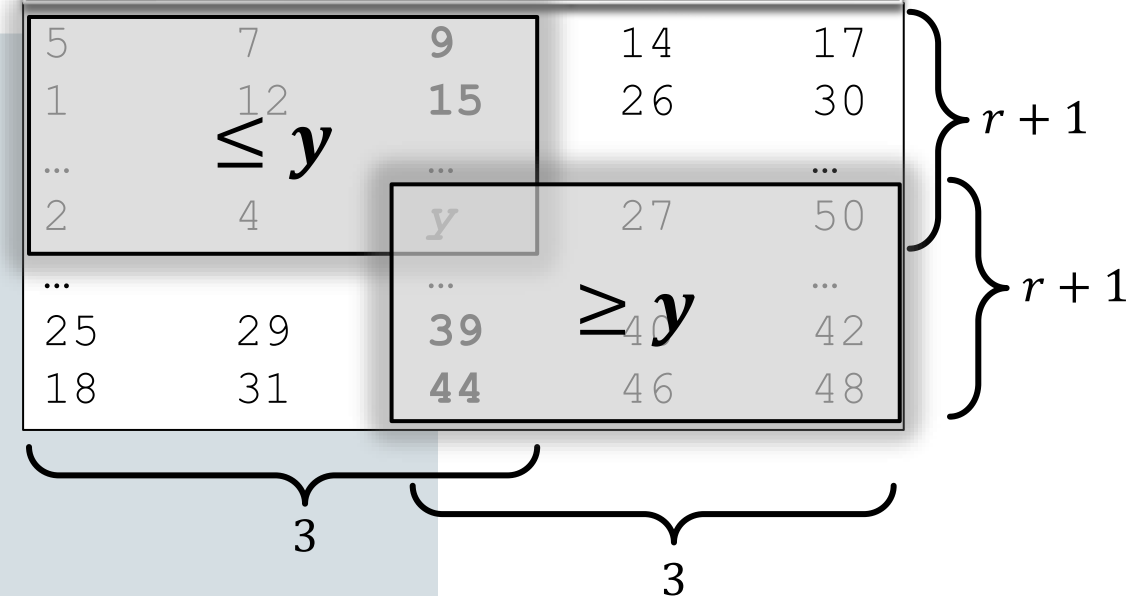
$T\left(\frac{n}{5}\right)$

$\Theta(n)$

$\Theta(1)$

$T(?)$

Rows $\boldsymbol{B}$ ordered by medians



5	7	9	14	17
1	12	15	26	30
…		…		…
2	4	y	27	50
…		…		…
25	29	39	40	42
18	31	44	46	48

$3(r+1)$ elements $\leq y$
$3(r+1)$ elements $\geq y$

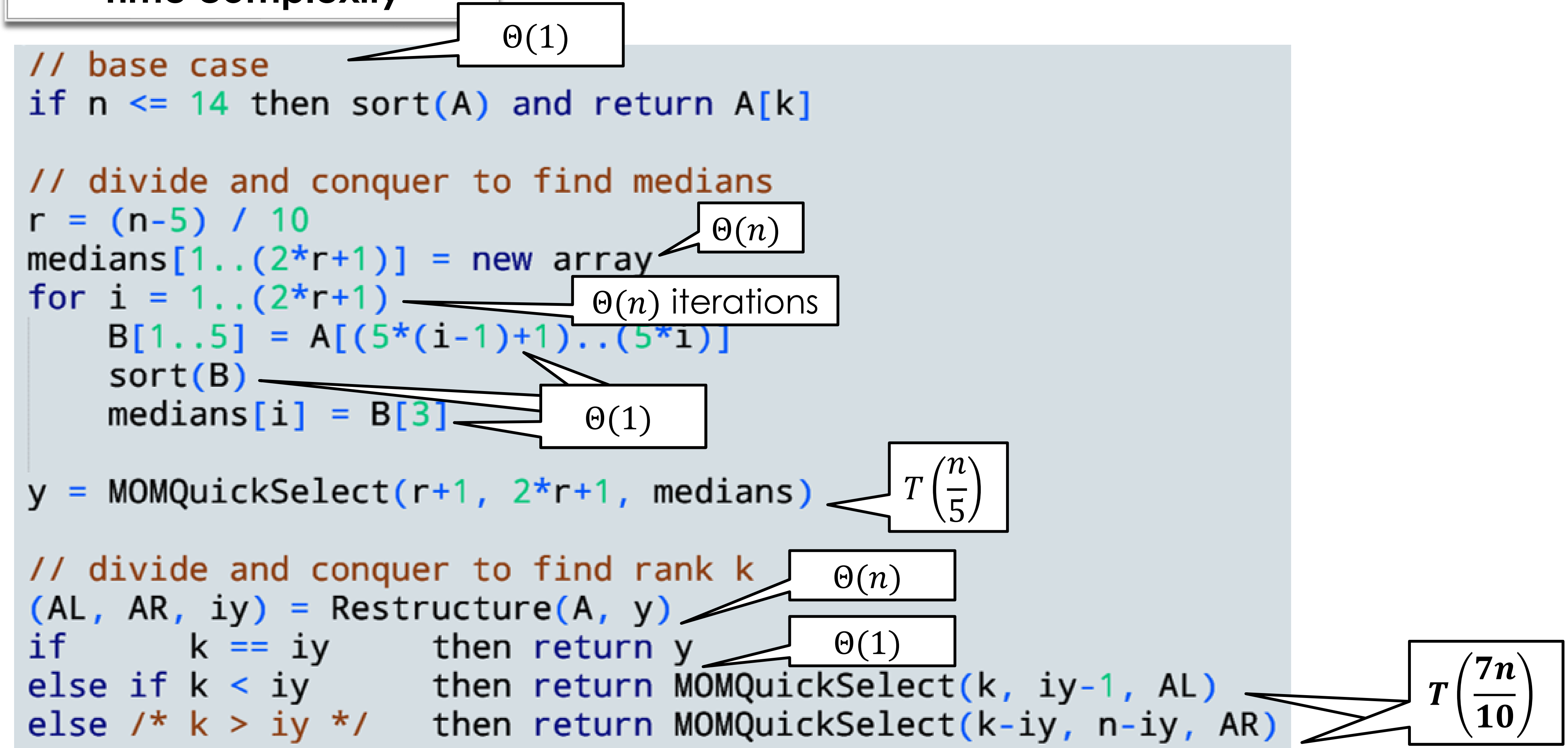
So problem size shrinks by at least $3(r+1)$

Observe $n = 10r + 5$

HOW MUCH DOES THE PROBLEM SHRINK?

- Shrinks by at least $3(r+1)$
- Problem size ~= $n = 10r + 5$
- Subproblem size $\leq n - Shrink = n - 3(r+1)$
  - $= 10r + 5 - 3r - 3 = 7r + 2$
  - Express in terms of n using $r = \left\lfloor \frac{n-5}{10} \right\rfloor$
    - Subproblem size $\leq 7\left\lfloor \frac{n-5}{10} \right\rfloor + 2 \leq 7\frac{n-5}{10} + 2$
    - $= \frac{7n}{10} - 7\left(\frac{5}{10}\right) + 2 = \frac{7n}{10} - \frac{3}{2} \leq \boldsymbol{\frac{7n}{10}}$

Time complexity



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    B[1..5] = A[(5*(i-1)+1)..(5*i)]
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y = MOMQuickSelect(r+1, 2*r+1, medians)

// divide and conquer to find rank k
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if      k == iy     then return y
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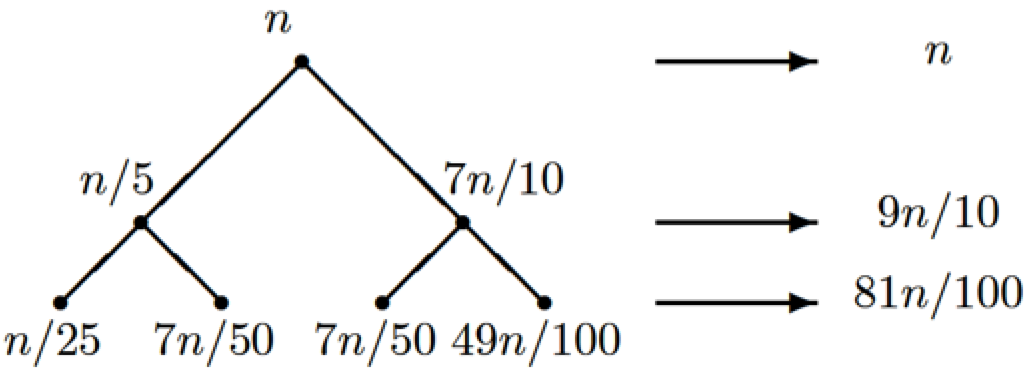
else /* k > iy */   then return MOMQuickSelect(k-iy, n-iy, AR)
```

$$T(n) \in O(n) + T(n/5) + T(7n/10) \qquad \text{if } n \geq 15$$
$$T(n) \in O(1) \qquad \text{if } n \leq 14$$

The key fact is that $1/5 + 7/10 = 9/10 < 1$.

$$\boxed{\begin{array}{ll} T(n) \in O(n) + T(n/5) + T(7n/10) & \text{if } n \geq 15 \\ T(n) \in O(1) & \text{if } n \leq 14 \end{array}}$$

The fact that $T(n) \in \Theta(n)$ can be proven formally using guess-and-check (induction) or informally using the recursion tree method.



etc.

$$\sum_{i=0}^{\infty} n \left(\frac{9}{10}\right)^i = 10n \in \Theta(n)$$

Guess & check
$\boldsymbol{T(n) = cn}$

- **Let** $\boldsymbol{T(n) = c'n + T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right)}$ **where** $\boldsymbol{c' > 0}$
- **Want to prove:** $\boldsymbol{T(n) = cn}$ **for some** $\boldsymbol{c > 0}$
- Note c and c' are independent constants
  - c' comes from the work at each level of recursion being $O(n)$
  - c is a positive constant we are trying to show exists
- **I.H.:** Suppose $\exists c > 0 : T(n') = cn'$ for $15 \leq n' < n$
- $T(n) = c'n + c\frac{n}{5} + c\frac{7n}{10}$ (by inductive hypoth.)
- $\boldsymbol{T(n) = cn}$ **(want this to be true)**
- $\Leftrightarrow\ c'n + c\frac{n}{5} + c\frac{7n}{10} = cn$ (equivalently)
- $\Leftrightarrow c' + c\frac{1}{5} + c\frac{7}{10} = c \Leftrightarrow c = 10c'$ (by algebra)