CS 341: ALGORITHMS

Lecture 5: finishing D&C, greedy algorithms I

Readings: see website

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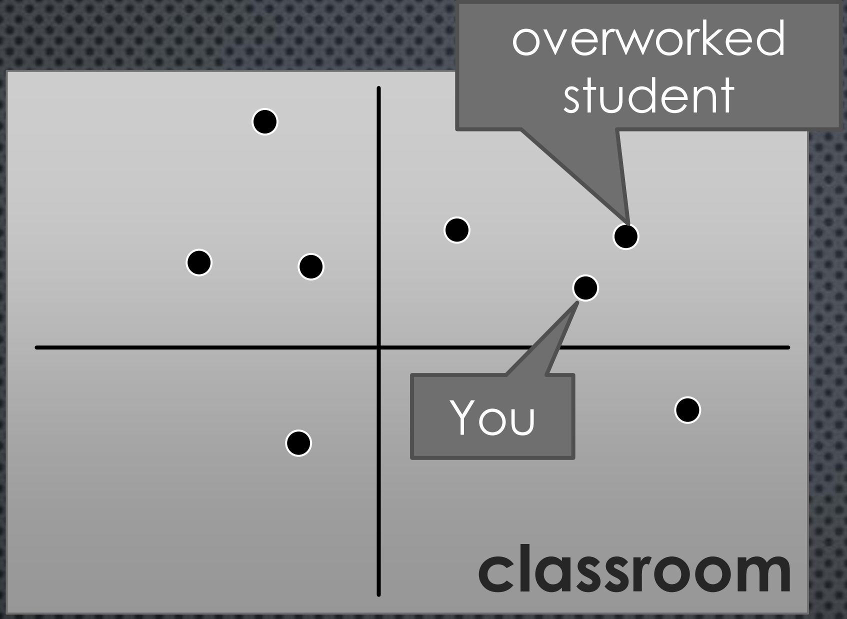
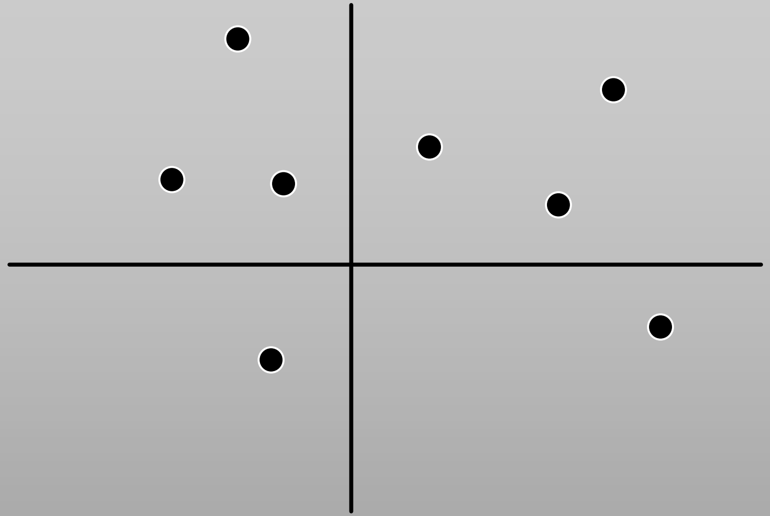
THE CLOSEST PAIR PROBLEM



THE **CLOSEST PAIR** PROBLEM

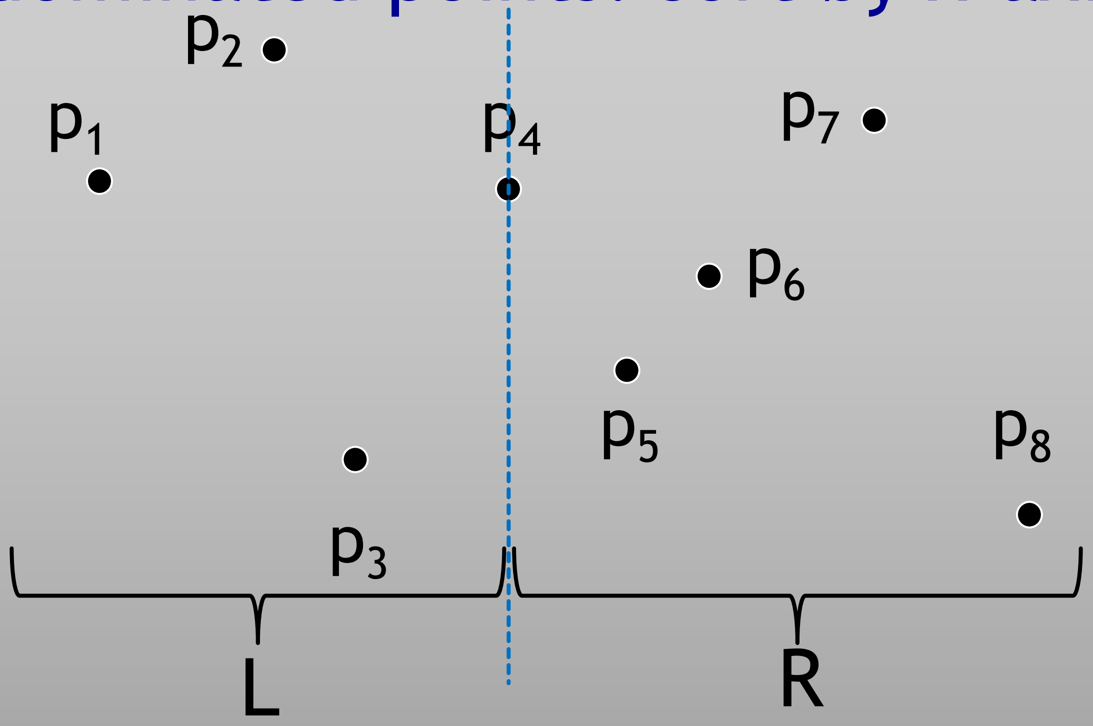
- Input: Set P of n 2D points



- Output: pair p and q s.t. dist(p, q) minimum over all pairs
- Break ties arbitrarily
- $dist(p,q) = \sqrt{(p.x - q.x)^2 + (p.y - q.y)^2}$

Can we Divide & Conquer?

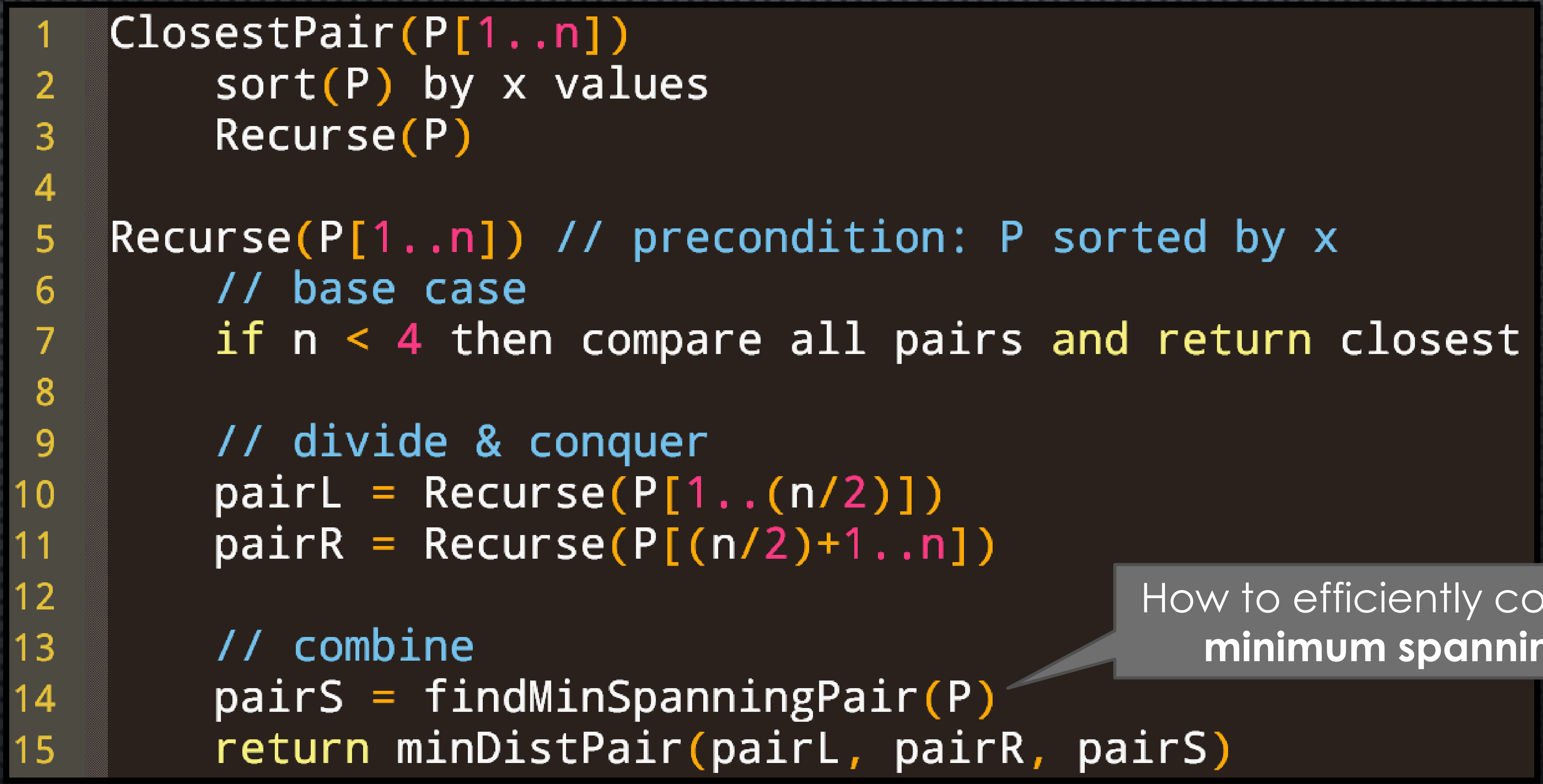
- Like non-dominated points: sort by x-axis & **divide in half**



Claim that doesn't require a proof: closest pair (p, q):

1. *(p, q) both in L or*
2. *(p, q) both in R or*
3. *One of (p,q) in L and one of (p,q) in R*

We call this a **spanning pair**



```
ClosestPair(P[1..n])
    sort(P) by x values
    Recurse(P)

Recurse(P[1..n]) // precondition: P sorted by x
    // base case
    if n < 4 then compare all pairs and return closest

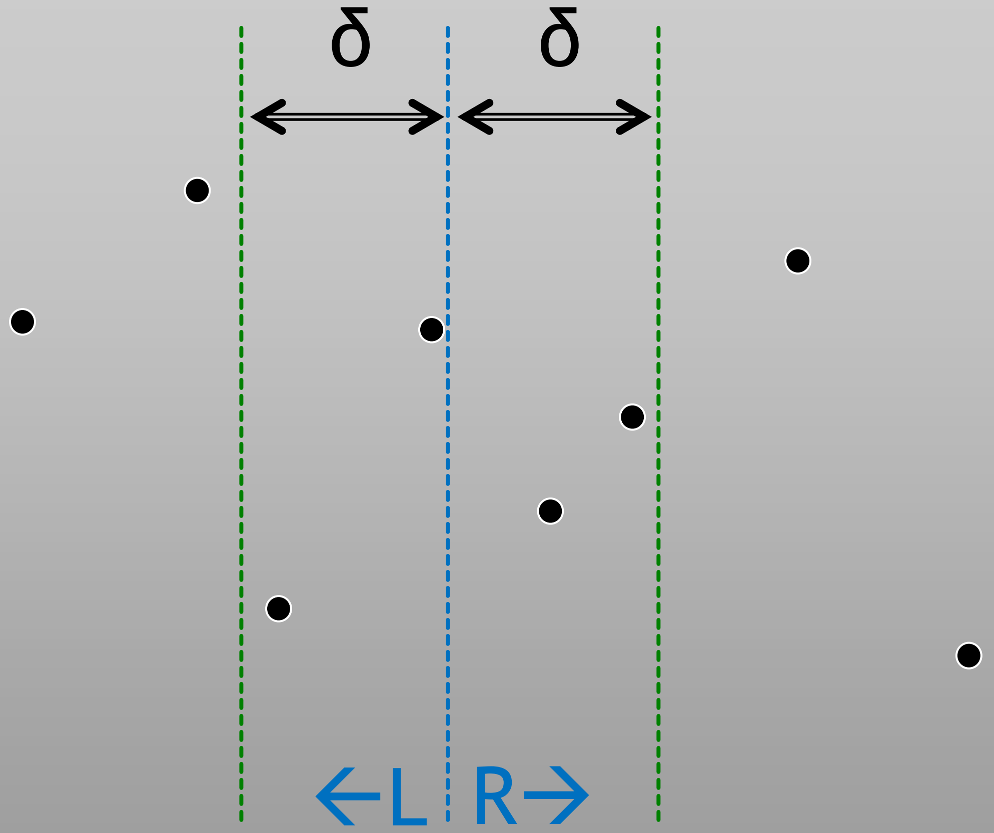
    // divide & conquer
    pairL = Recurse(P[1..(n/2)])
    pairR = Recurse(P[(n/2)+1..n])

    // combine
    pairS = findMinSpanningPair(P)
    return minDistPair(pairL, pairR, pairS)
```

How to efficiently compute the **minimum spanning pair**?

Observation 1

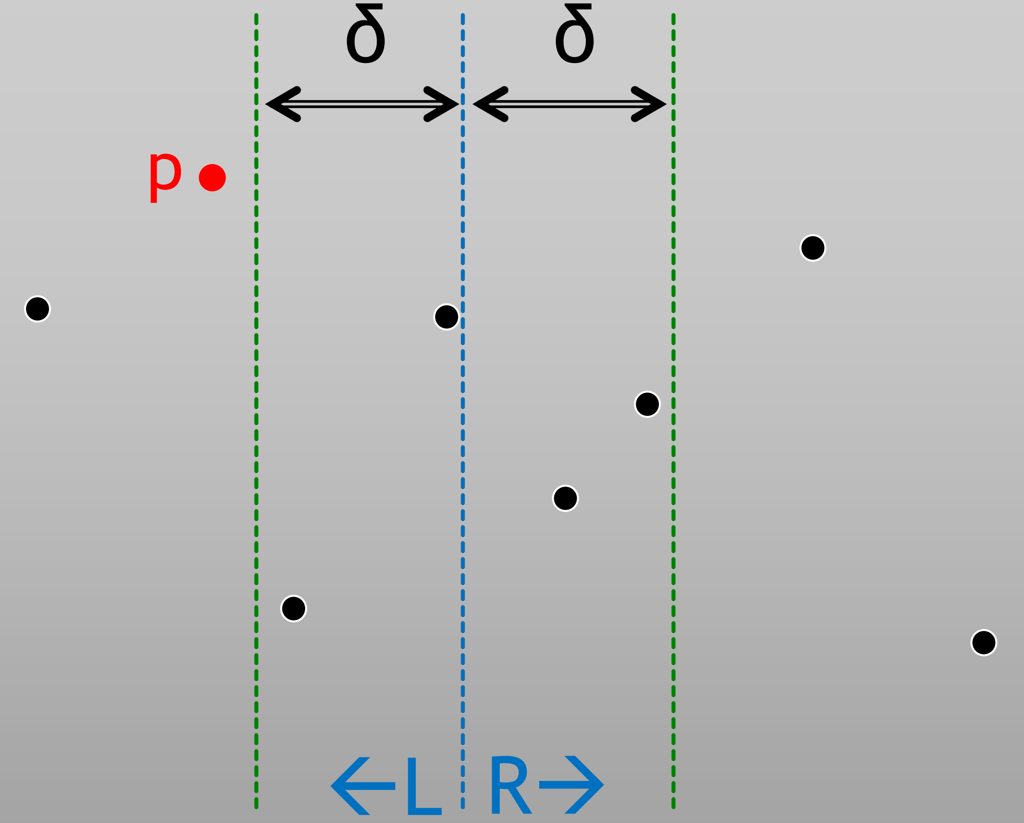
- Let $\delta = \min(\text{dist}(\text{pair}_L), \text{dist}(\text{pair}_R))$



- Then pair_s (if closest globally) lies in the above 2δ-wide green strip

Q: Why?

Example for Observation 1



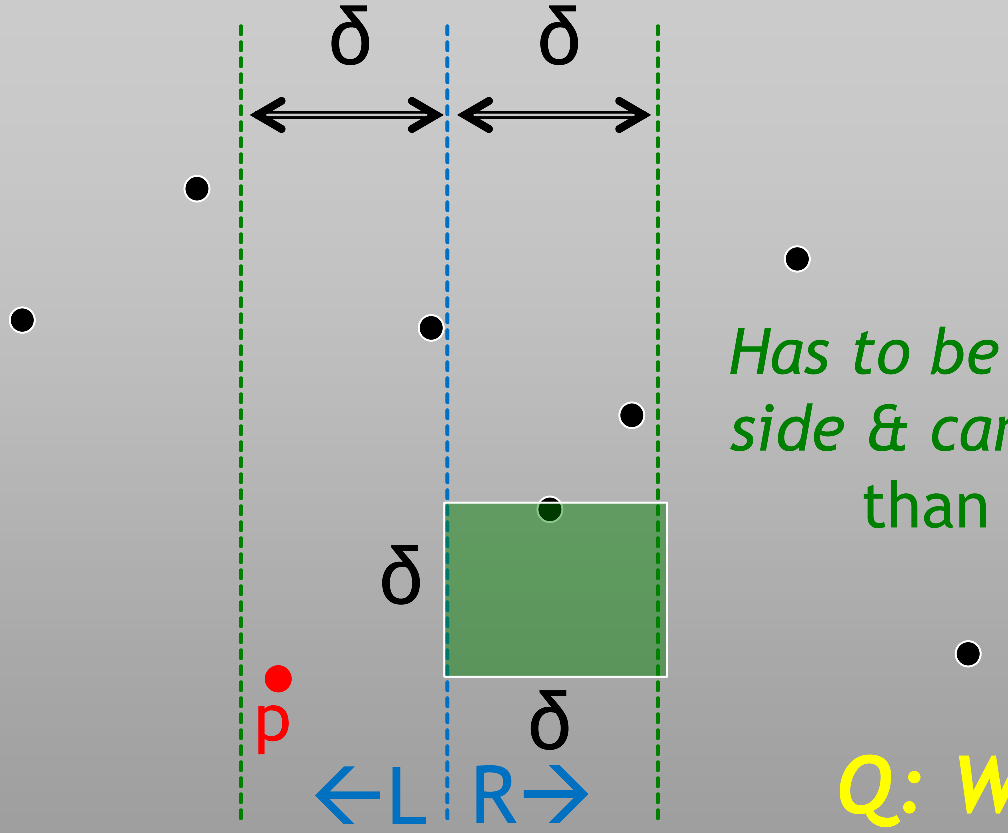
Q: Can p be part of a globally closest spanning pair$_s$?

A: No. ***Everything*** *in R has* ***dist > δ*** *to p.*

And we already have a solution with dist = ***δ.***

Observation 2

- Say, p (the lowest y valued point in strip) is in pair_s



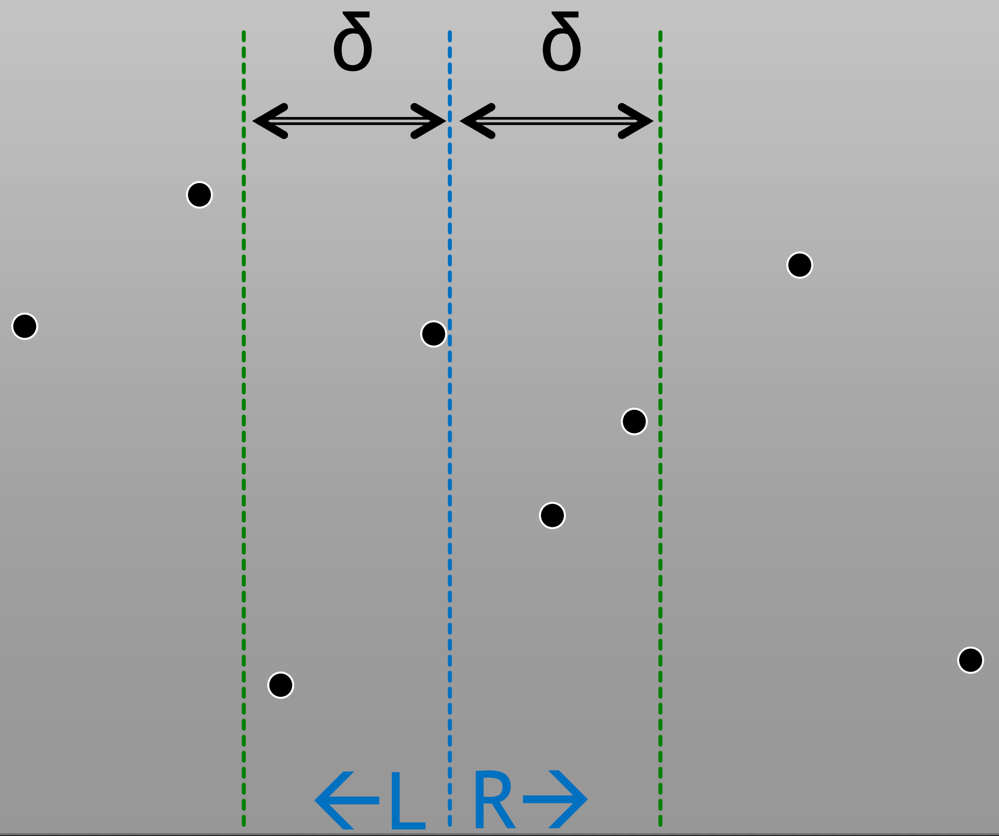
Has to be on the opposite side & can't be > δ higher than p on y axis.

Q: Why?

- Then the other point can only lie in this δxδ square.

Core Idea For Finding Spanning Pair

1. Start from lowest y valued point in the strip
2. Search the δxδ square points on the opposite side
3. Repeat 1 & 2 for the next lowest y-valued point
4. So on and so forth…



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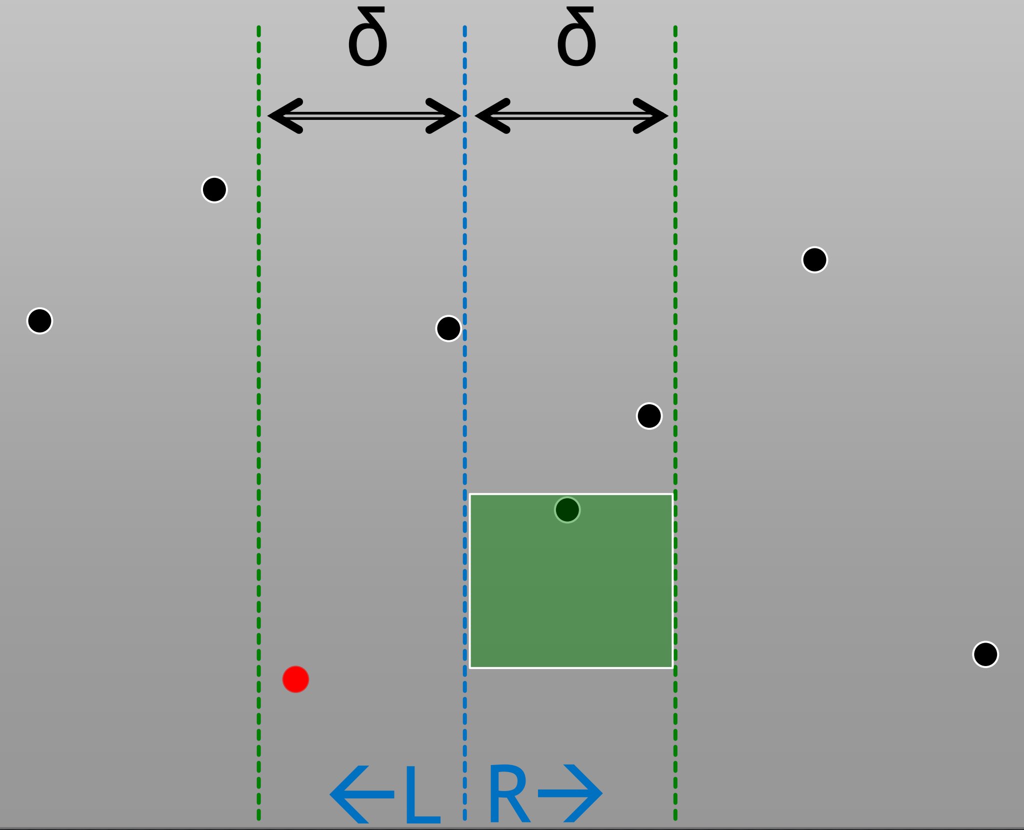

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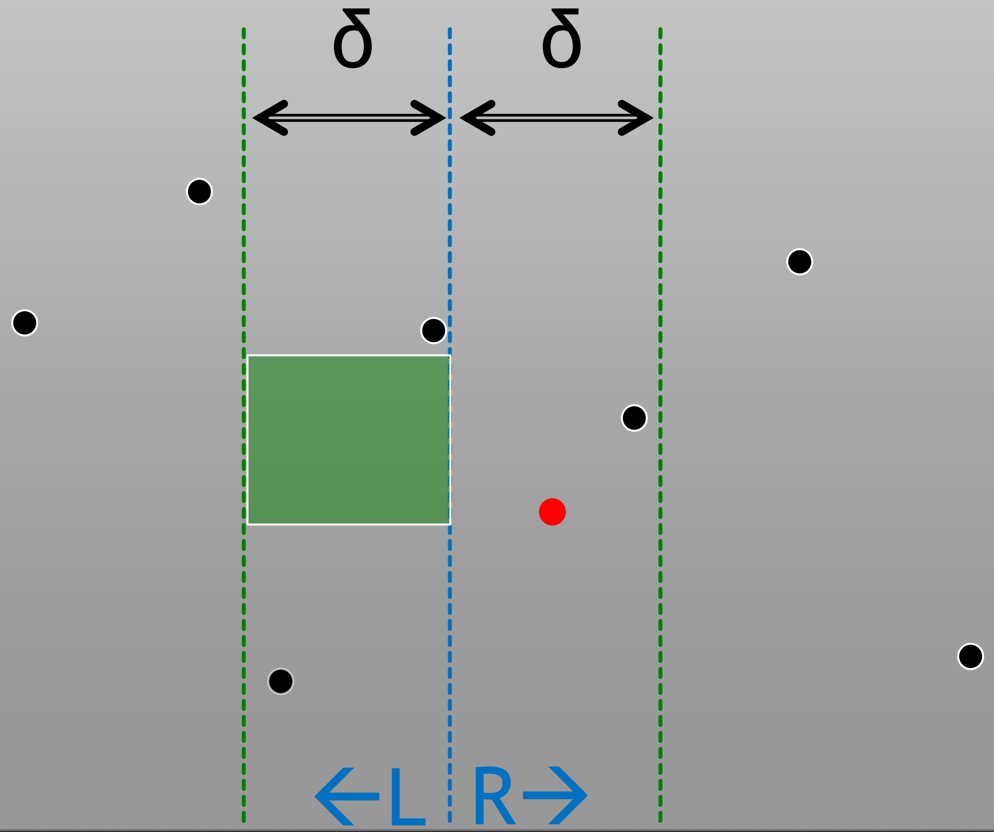

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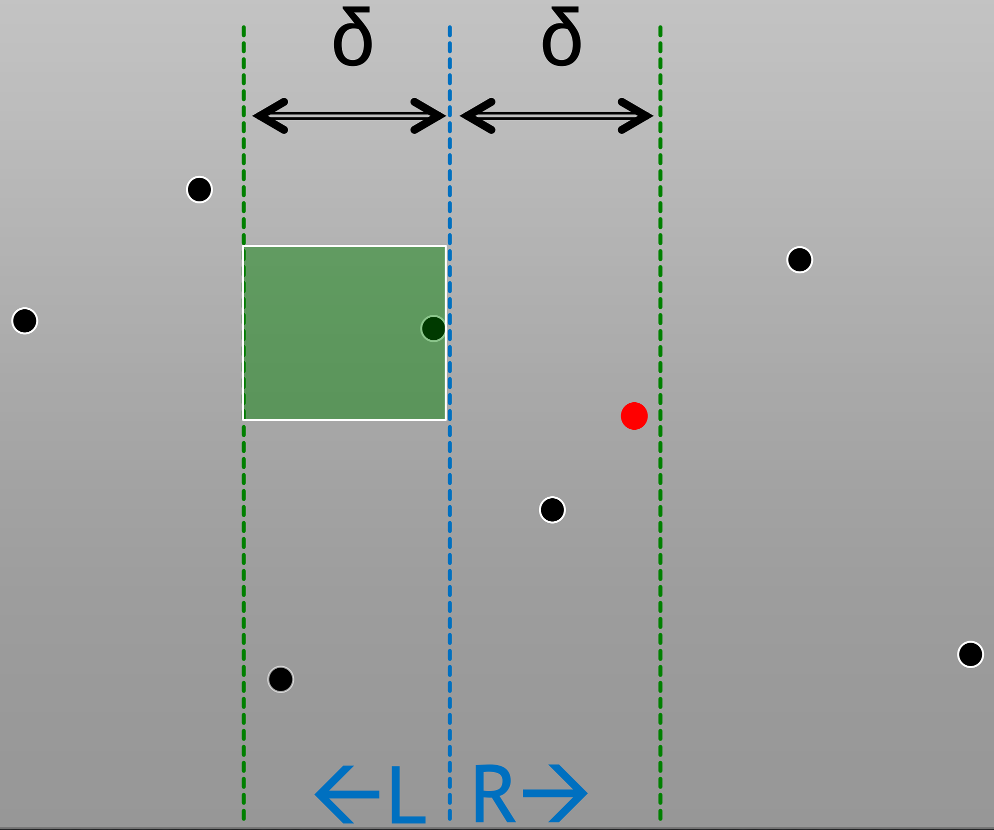

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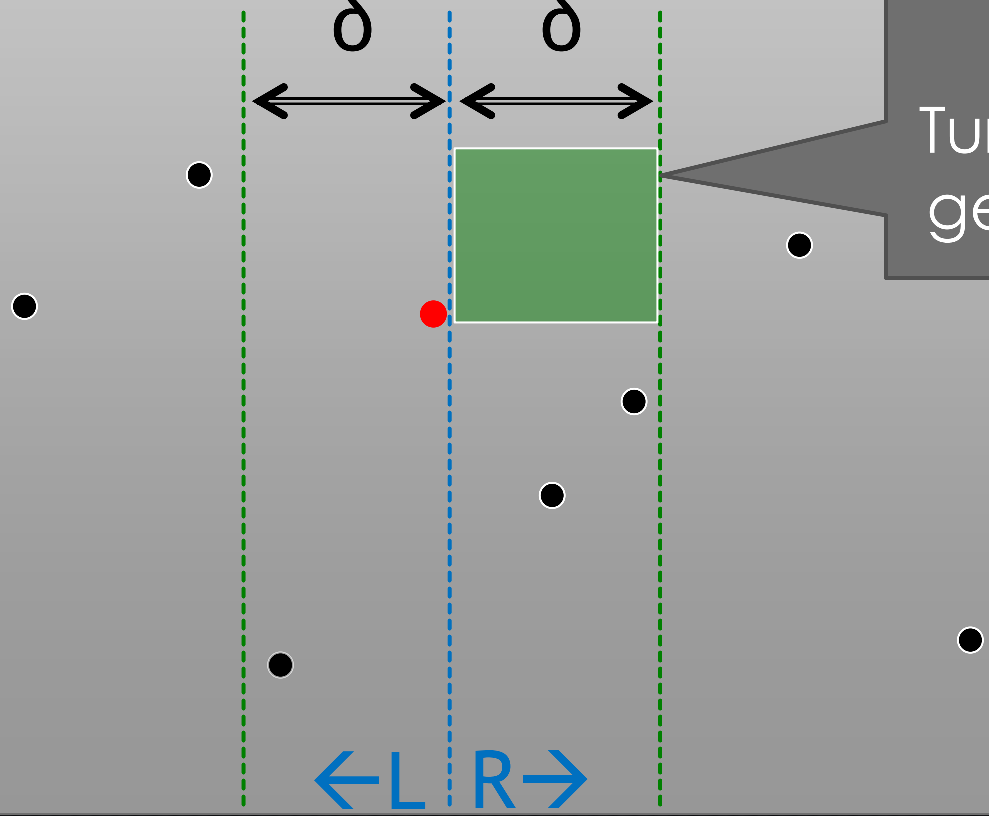
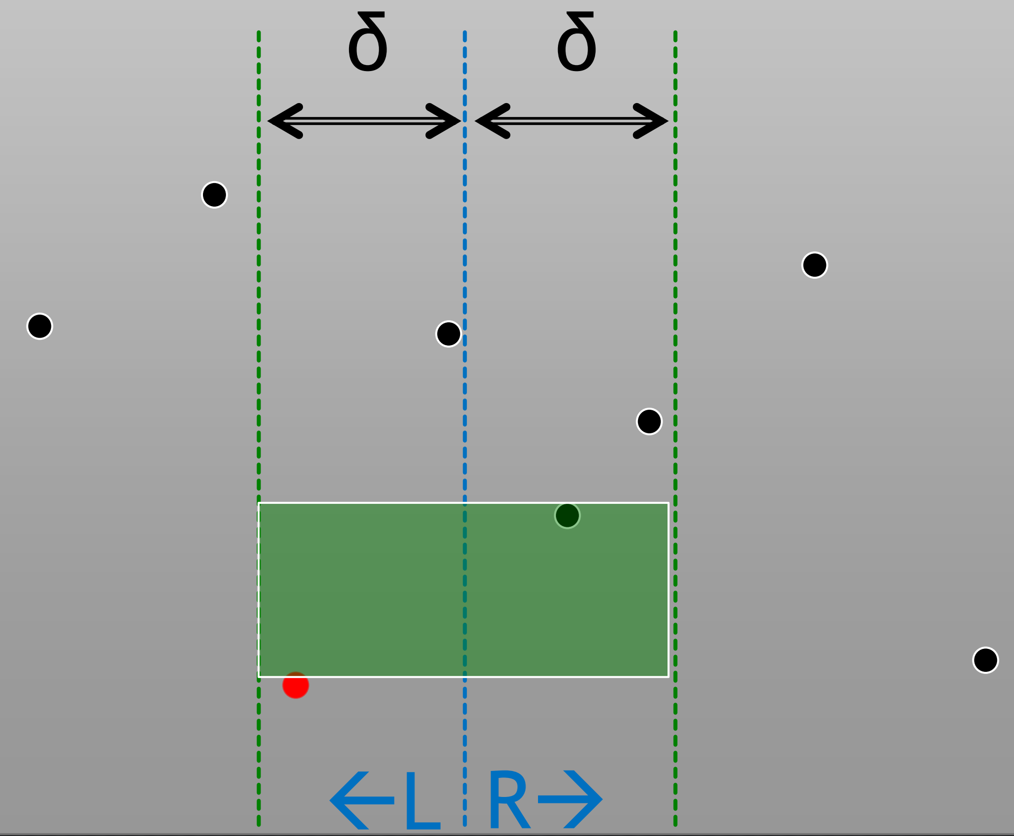


Switching sides might complicate code…
Turns out it's not needed to get good time complexity.

A More Practical Idea

- Don’t differentiate between same and opposite side
- Just search the 2δxδ above rectangle each time



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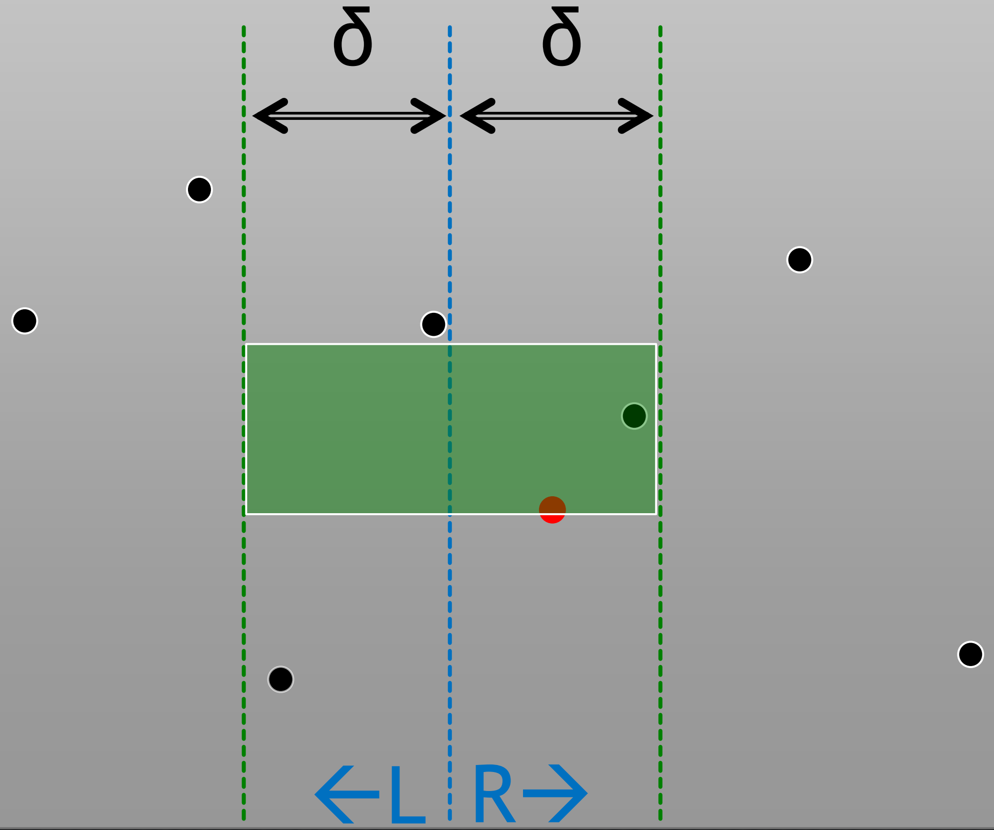

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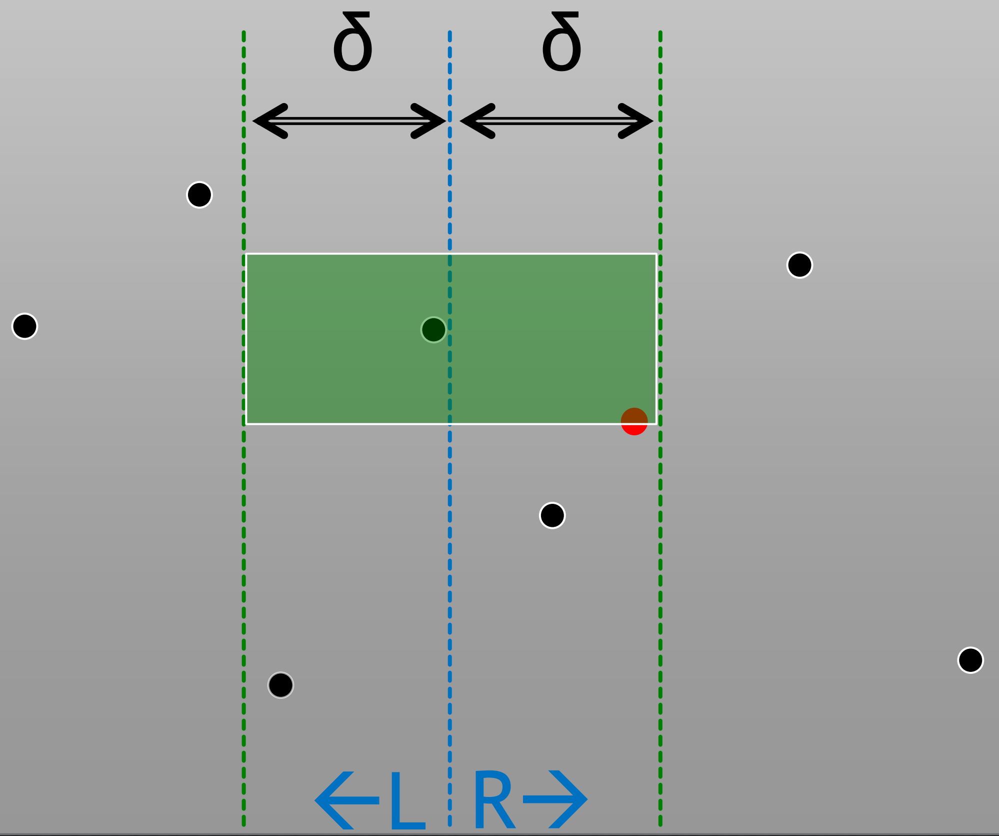

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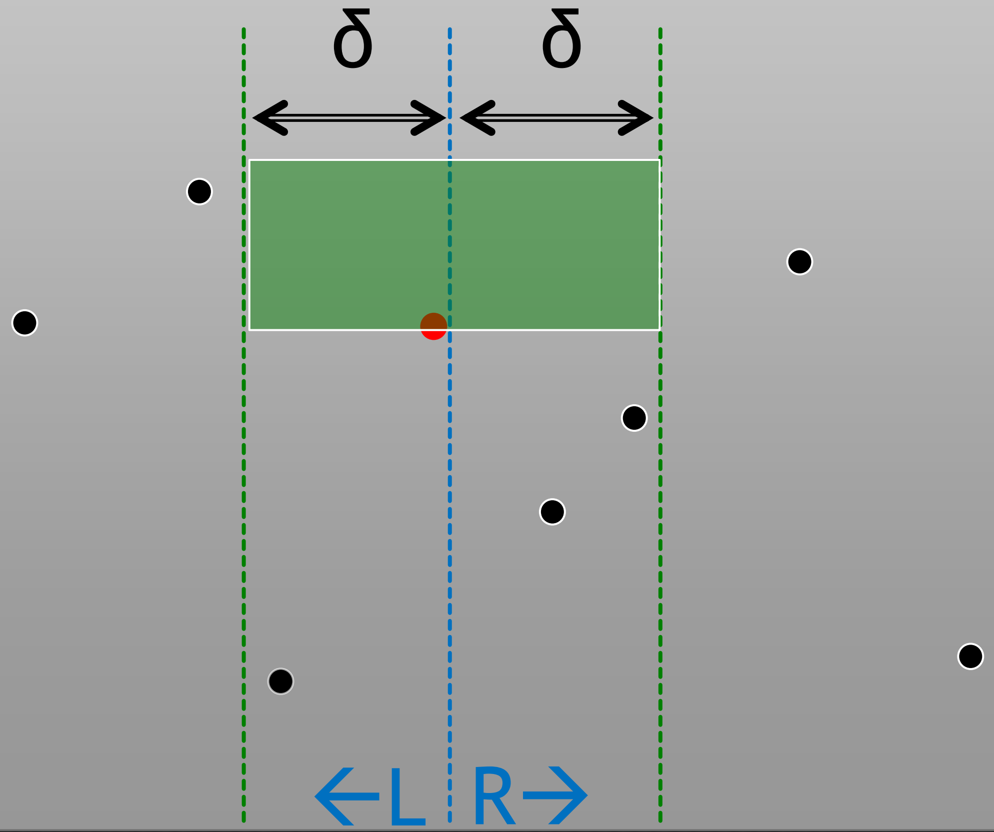

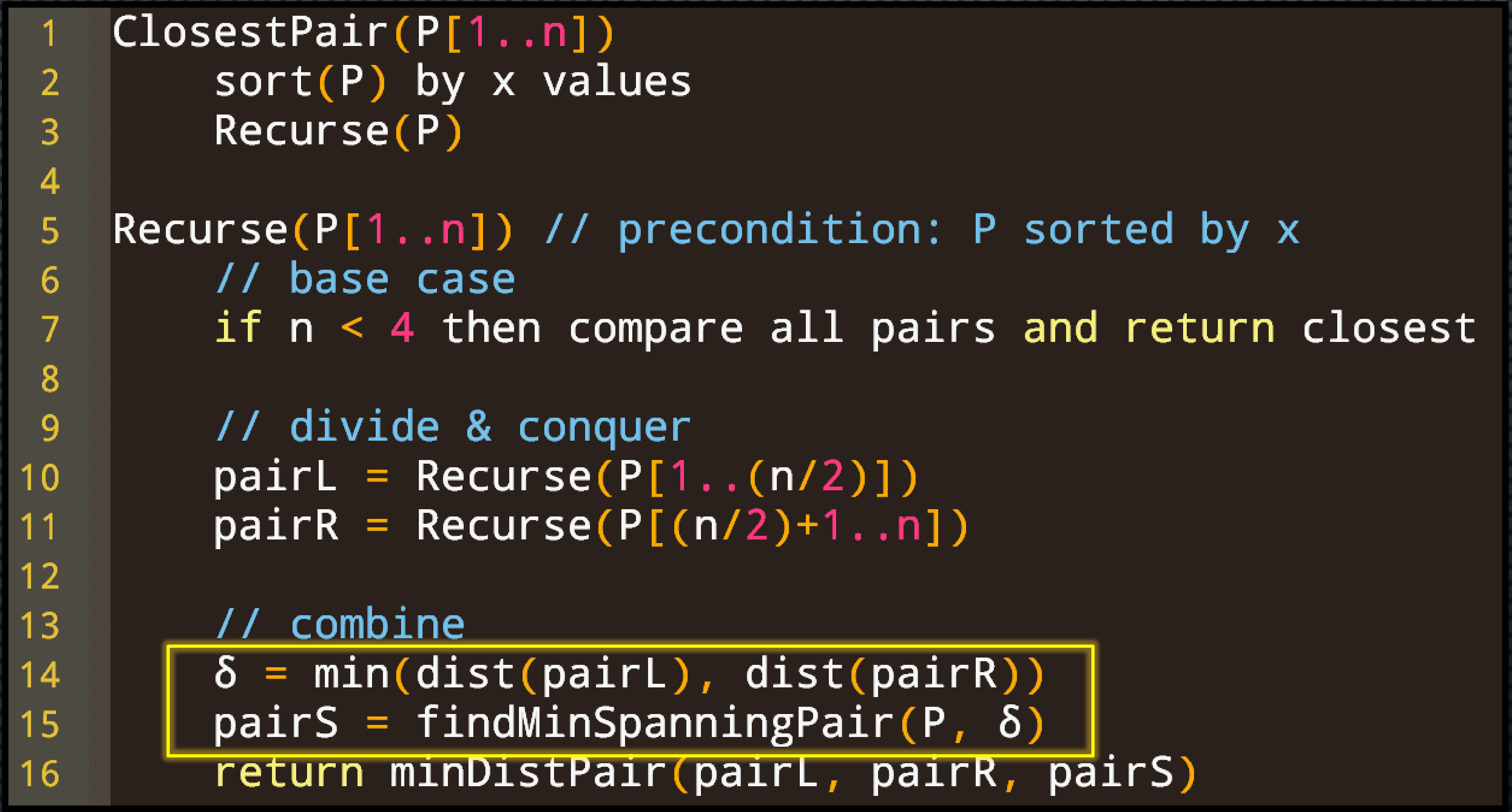

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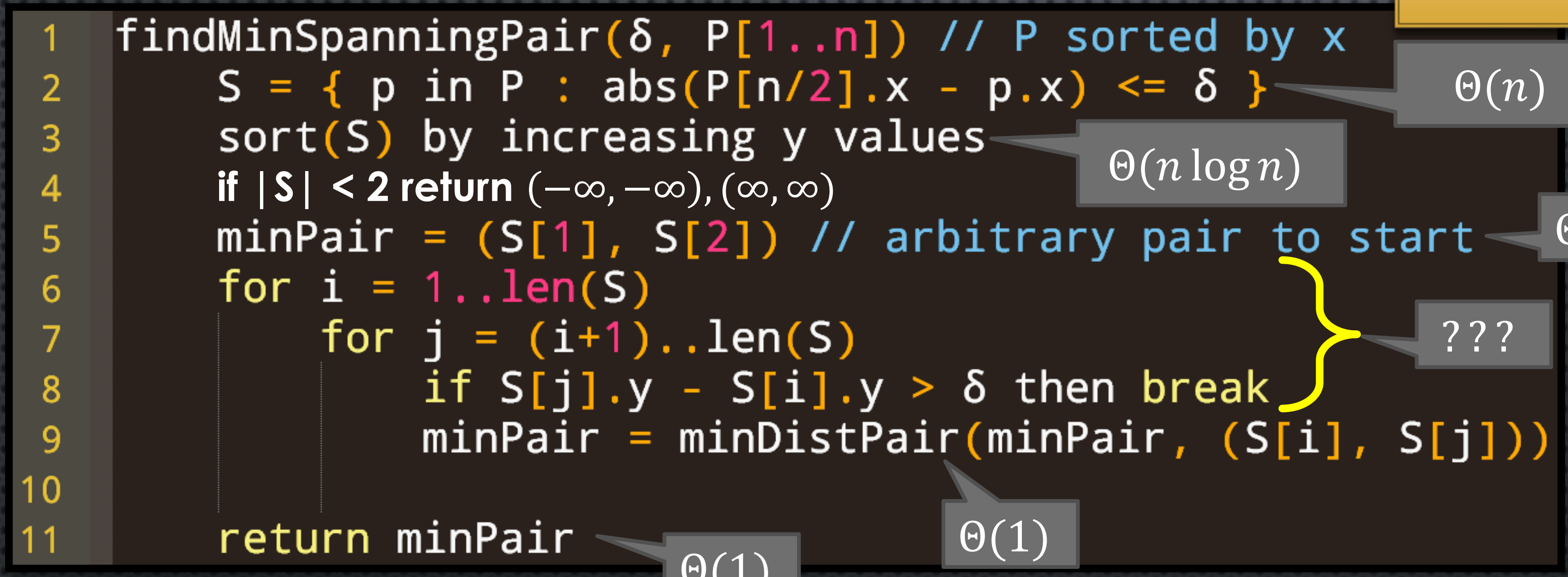
Recurse(P[1..n]) // precondition: P sorted by x
    // base case
    if n < 4 then compare all pairs and return closest

    // divide & conquer
    pairL = Recurse(P[1..(n/2)])
    pairR = Recurse(P[(n/2)+1..n])

    // combine
    δ = min(dist(pairL), dist(pairR))
    pairS = findMinSpanningPair(P, δ)
    return minDistPair(pairL, pairR, pairS)
```

Time complexity?



```
findMinSpanningPair(δ, P[1..n]) // P sorted by x
    S = { p in P : abs(P[n/2].x - p.x) <= δ }
    sort(S) by increasing y values
    if |S| < 2 return $(-\infty,-\infty),(\infty,\infty)$
    minPair = (S[1], S[2]) // arbitrary pair to start
    for i = 1..len(S)
        for j = (i+1)..len(S)
            if S[j].y - S[i].y > δ then break
            minPair = minDistPair(minPair, (S[i], S[j]))

    return minPair
```

$\Theta(n)$

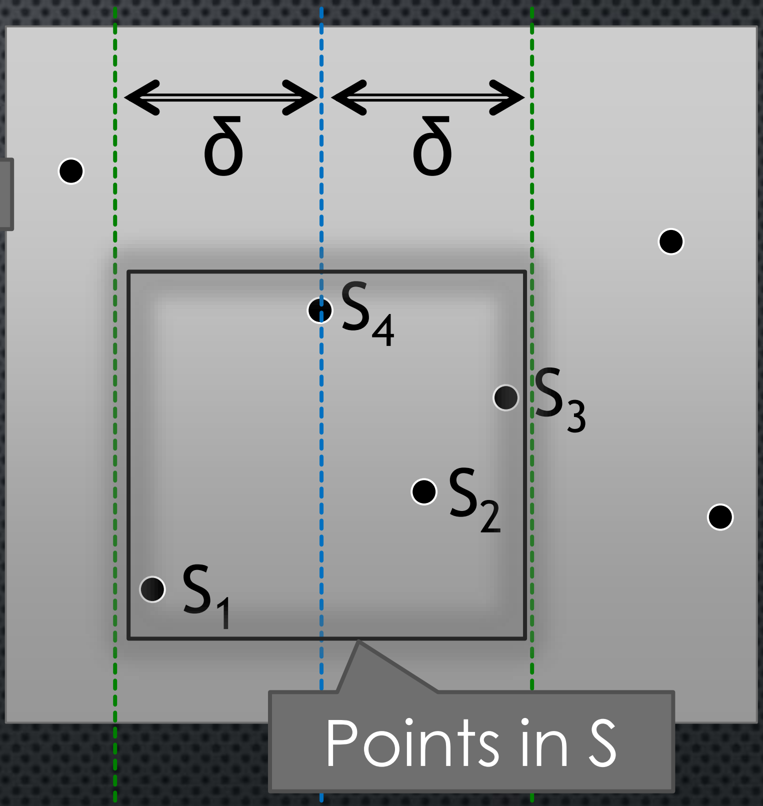
$\Theta(n \log n)$

$\Theta(1)$

???

$\Theta(1)$

$\Theta(1)$



*Claim: inner loop performs **O(1)** iterations!*

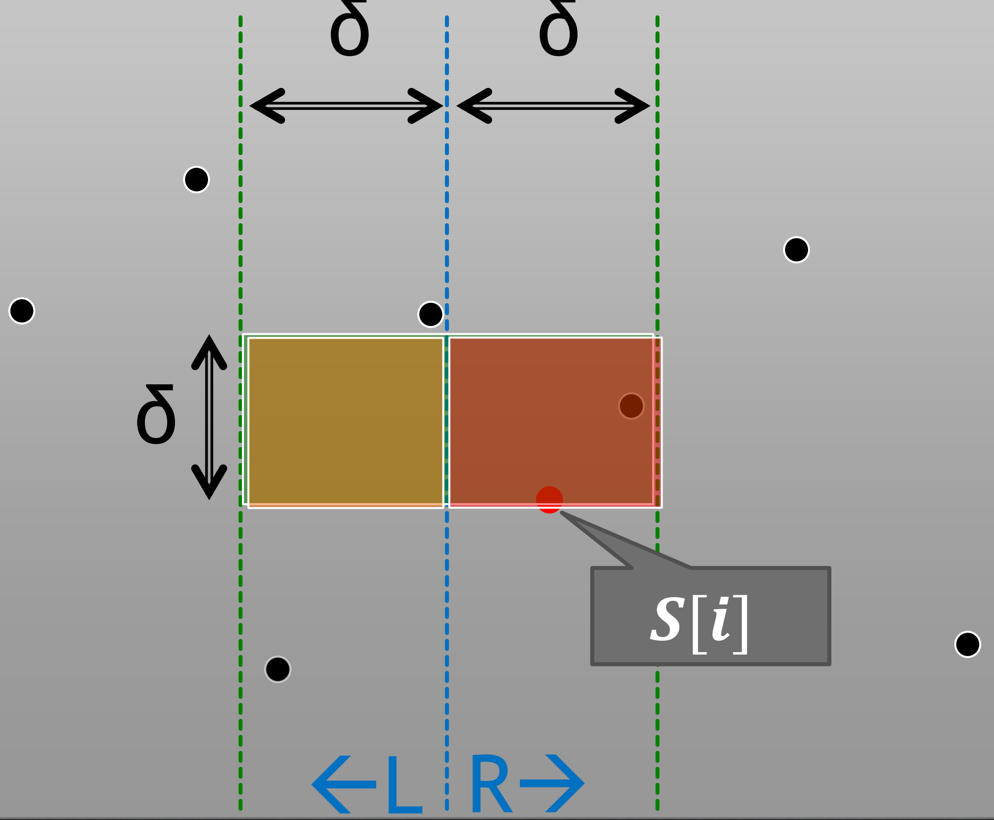
For a particular *i*,
how many *j* iterations occur?

```
for i = 1..len(S)
    for j = (i+1)..len(S)
        if S[j].y - S[i].y > δ then break
```

Obs: as many as there are **points** in the 2δ x δ rectangle.

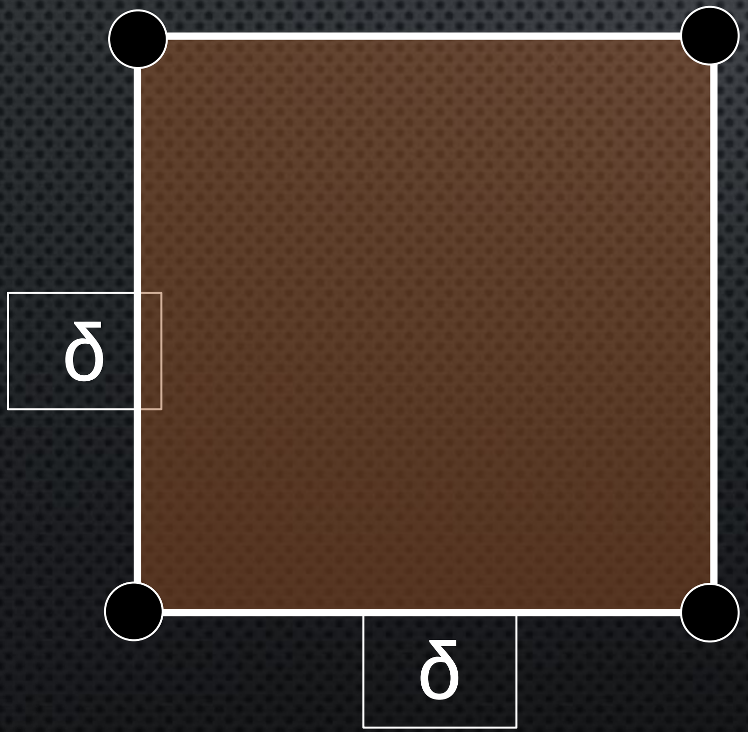
Q: How many points can be in a 2δ x δ rectangle?

A: As many as in the left δ x δ square + right δ x δ square.



POINTS IN A $\delta \times \delta$ SQUARE

- Recall δ is the smallest distance between any pair of points that are both in L or both in R
- Note this square is entirely in L or entirely in R



So, δ is the smallest distance between any pair of points in this square!

A point in the middle would rule out any other points

So, most efficient packing of points puts **one in each corner** (4 total)

For a particular i,
how many j iterations occur?

```
for i = 1..len(S)
    for j = (i+1)..len(S)
        if S[j].y - S[i].y > δ then break
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Obs: as many as there are **points** in the 2δ x δ rectangle.

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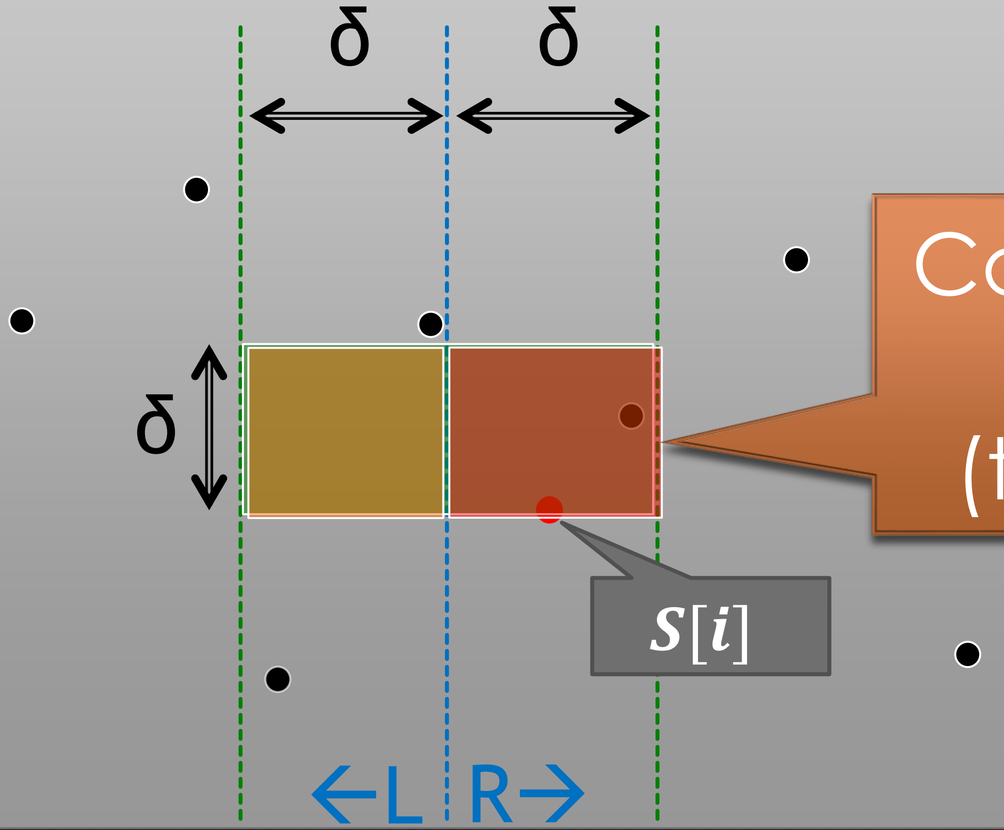
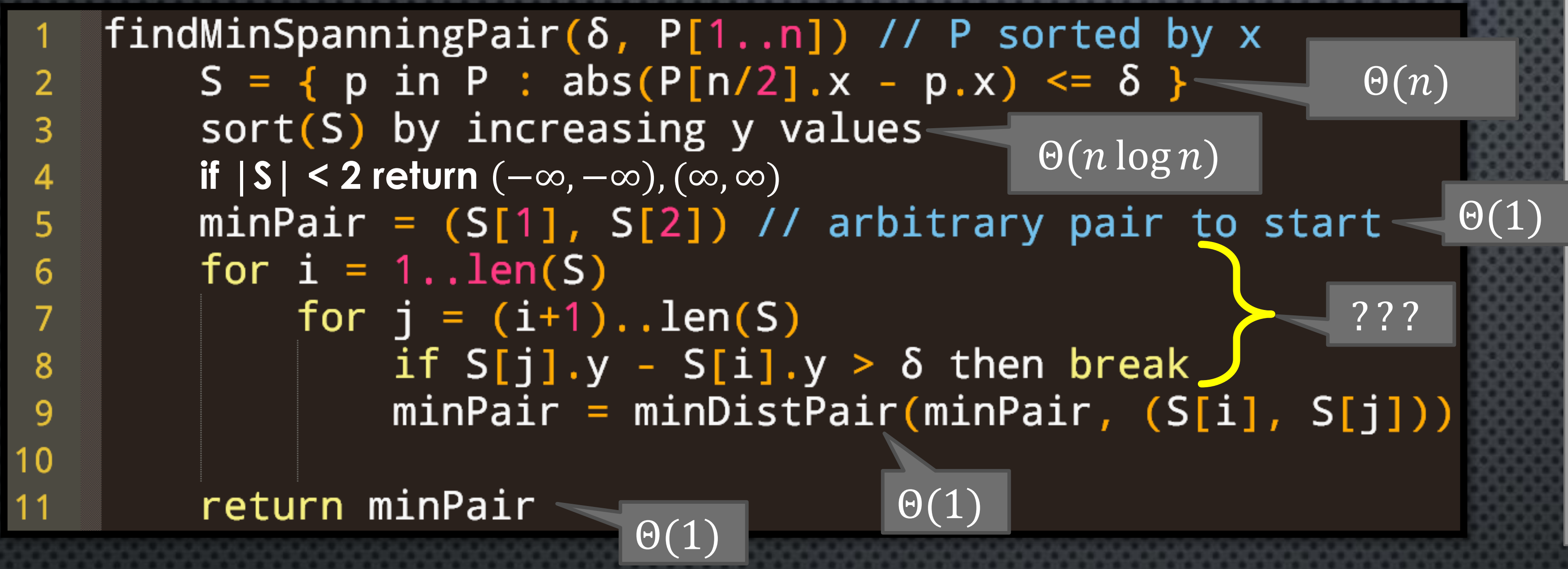


Can only contain **eight** points!
(technically six)

Time complexity (unit cost)



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    minPair = (S[1], S[2]) // arbitrary pair to start
    for i = 1..len(S)
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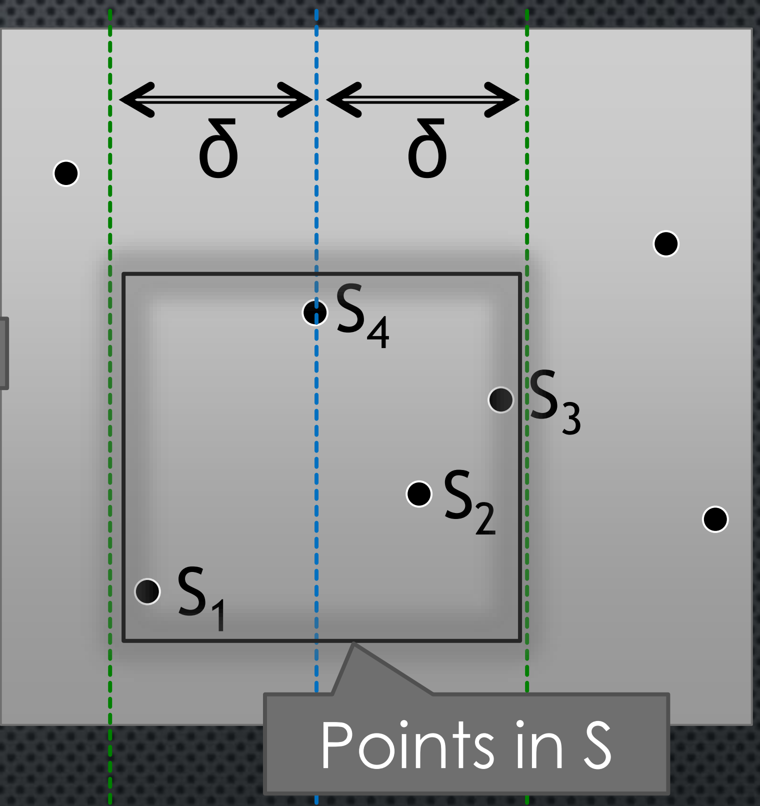
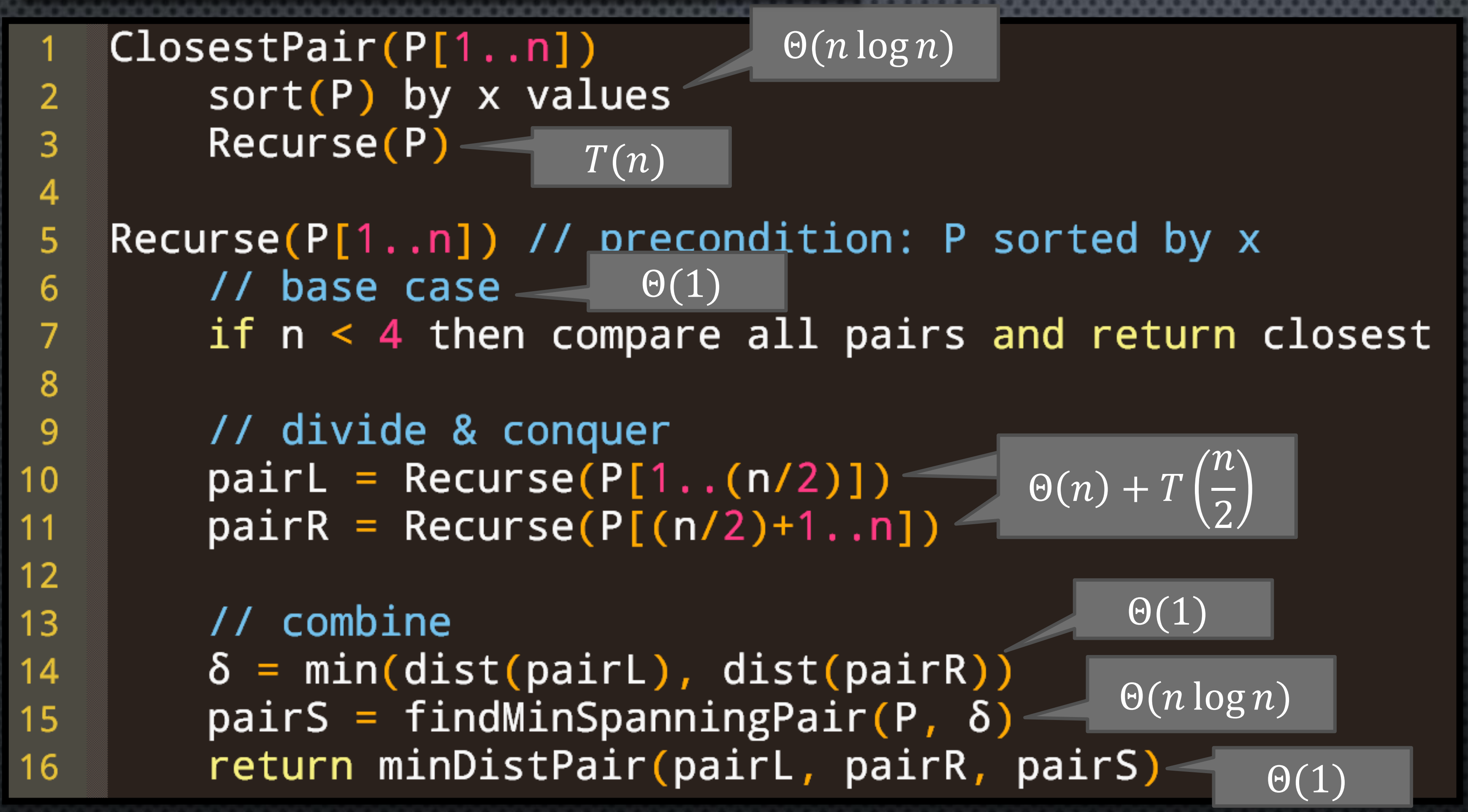
            minPair = minDistPair(minPair, (S[i], S[j]))

    return minPair
```



- *j*-loop performs at most **eight** iterations
- Each does $\Theta(1)$ work, so entire *j*-loop does $\Theta(1)$ work!
- So entire *i*-loop does $\Theta(n)$ work
- So, findMinSpanningPair does $\boldsymbol{\Theta(n \log n)}$ work

Time complexity (unit cost)



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```

$\Theta(n \log n)$

$T(n)$

$\Theta(1)$

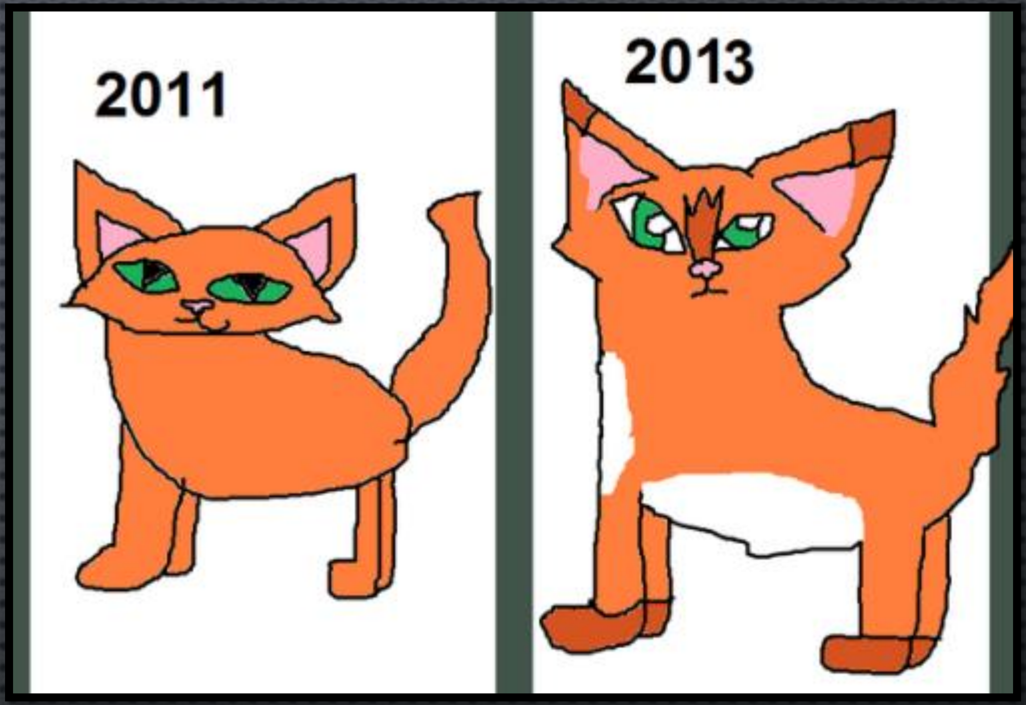
$\Theta(n) + T\left(\frac{n}{2}\right)$

$\Theta(1)$

$\Theta(n \log n)$

$\Theta(1)$

- $T'(n)$: $ClosestPair(P[1..n])$
- $T(n)$: $Recurse(P[1..n])$
- $T'(n) = \Theta(n \log n) + T(n)$
- $T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n \log n)$
- Lec2 notes using recursion trees showed
  - $\boldsymbol{T(n) \in \Theta(n \log^2 n)}$
- $T'(n) \in \Theta(n \log n) + \Theta(n \log^2 n)$
- **So $\boldsymbol{T'(n) \in \Theta(n \log^2 n)}$**



IMPROVING THIS RESULT FURTHER

IMPROVING THE PREVIOUS ALGORITHM

- Sorting by y-values causes findMinSpanningPair to take $O(n \log n)$ time instead of $O(n)$ time
- This happens in each recursive call,
  and dominates the running time
- Avoid sorting P over and over by creating
  another copy of P that is **pre-sorted by y-values**
- Assume for simplicity that x coordinates are unique

Shamos' algorithm (1975)

```
ShamosClosestPair(P[1..n])
    Px = sort(P) by increasing x values
    Py = sort(P) by increasing y values
    Recurse(Px, Py)

Recurse(Px[1..n], Py[1..n])
    // base case
    if n < 4 then return BruteForce(Px)

    // divide & conquer
    xmid = Px[n/2].x
    PxL = Px[1..(n/2)]              // x <= xmid
    PxR = Px[(n/2+1)..n]            // x >  xmid
    PyL = select p from Py where p.x <= xmid
    PyR = select p from Py where p.x >  xmid
    pairL = Recurse(PxL, PyL)
    pairR = Recurse(PxR, PyR)

    // combine
    δ = min(dist(pairL), dist(pairR))
    pairS = findMinSpanningPair(δ, Py, xmid)
    return minDistPair(pairL, pairR, pairS)
```

This selection step
preserves the y-sort order

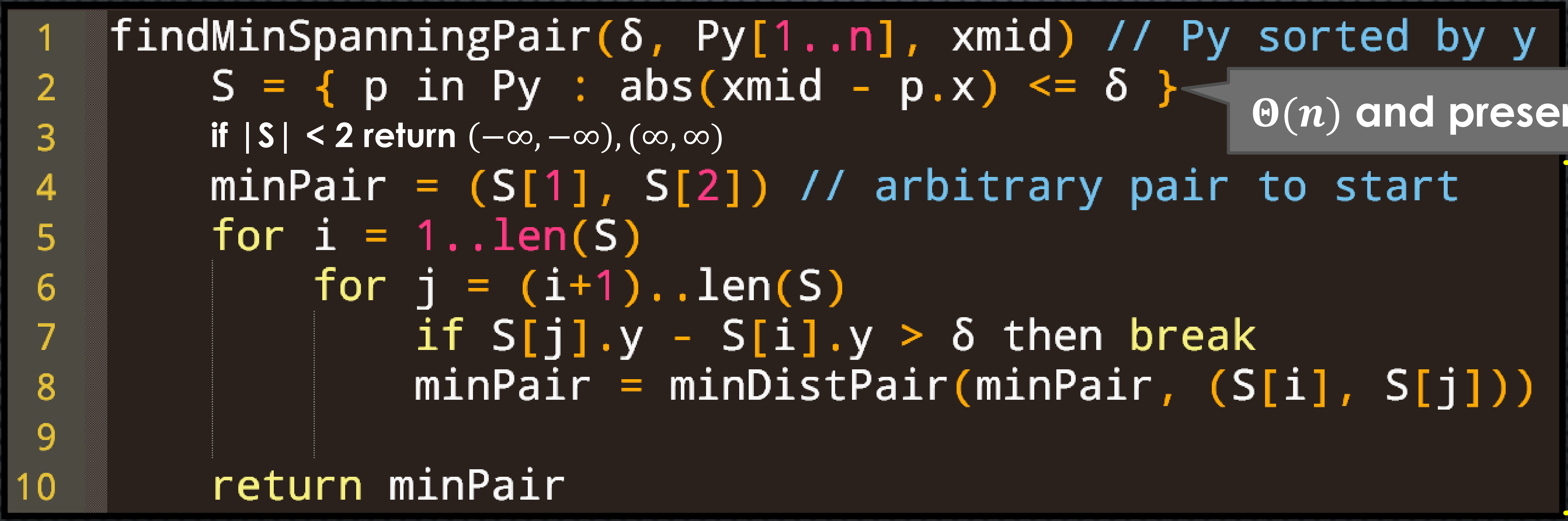
x-coord
uniqueness used

Observe PxL and PyL contain the **same points**

(specifically the points with x <= xmid)

Moreover PxL is sorted by x while PyL is sorted by y

And similarly for PxR, PyR…
No need to sort in Recurse!



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findMinSpanningPair(δ, Py[1..n], xmid) // Py sorted by y
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    minPair = (S[1], S[2]) // arbitrary pair to start
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            if S[j].y - S[i].y > δ then break
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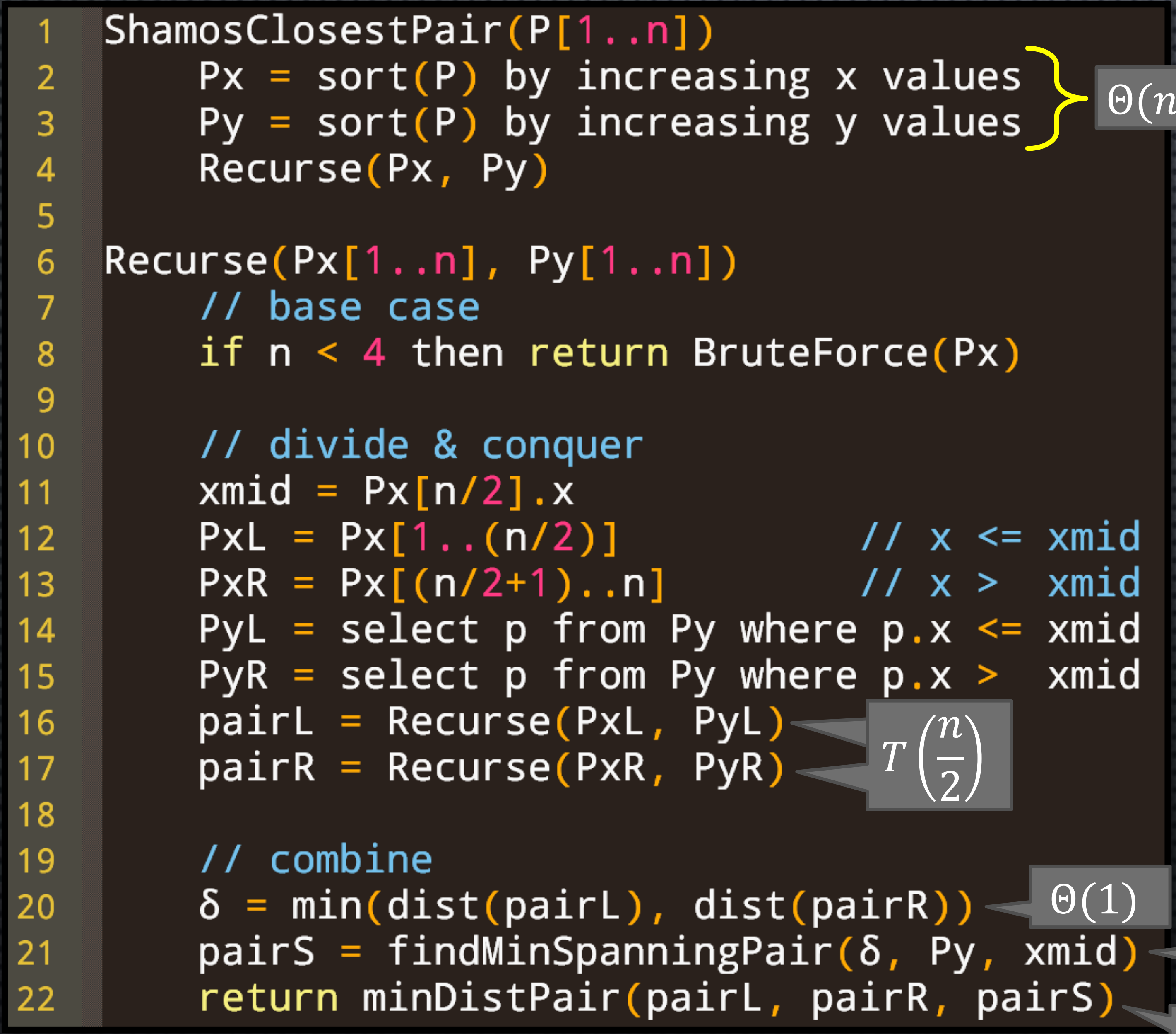
$\Theta(n)$ and preserves the y-sort order

$\Theta(n)$

Total $\Theta(n)$ for this function

Time complexity



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$\Theta(n \log n)$

$\Theta(n)$

$T\left(\frac{n}{2}\right)$

$\Theta(1)$

$\Theta(n)$

$\Theta(1)$

$$T(n) = 2T\left(\frac{n}{2}\right) + \Theta(n)$$

Merge sort recurrence…
$T(n) \in \Theta(n \log n)$

So runtime for Shamos' algorithm is in $\boldsymbol{\Theta(n \log n)}$

GREEDY ALGORITHMS

Optimization Problems

Problem: Given a problem instance, find a feasible solution that maximizes (or minimizes) a certain objective function.

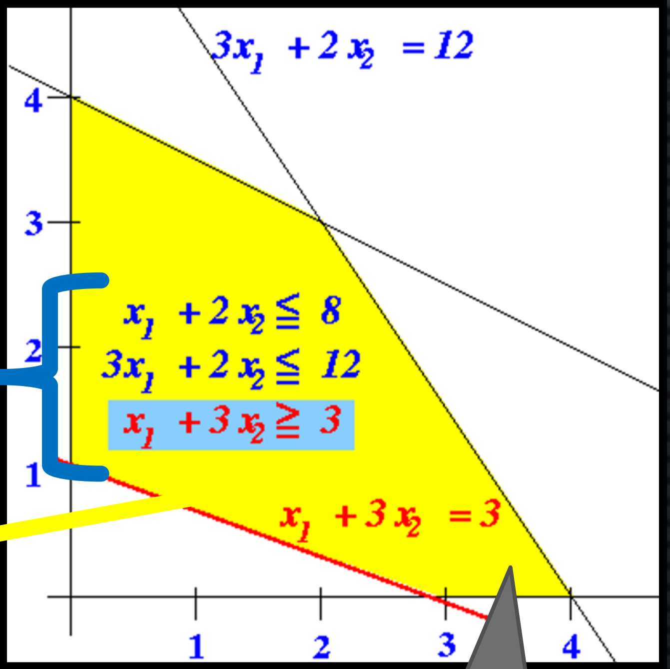
Problem Instance: **Input** for the specified problem.

Problem Constraints: **Requirements** that must be satisfied by any feasible solution.

Feasible Solution: For any problem instance I, *feasible*(I) is the set of all outputs (i.e., solutions) for the instance I that satisfy the given constraints.

Objective Function: A function $f : \mathit{feasible}(I) \to \mathbb{R}^+ \cup \{0\}$. We often think of f as being a **profit** or a **cost** function.

Optimal Solution: A feasible solution $X \in \mathit{feasible}(I)$ such that the profit $f(X)$ is maximized (or the cost $f(X)$ is minimized).



f(this point) = $720

SOLVING OPTIMIZATION PROBLEMS

- Lots of techniques
- We will study **greedy** approaches first
- Later, dynamic programming
  - Sort of like divide and conquer
    but *can* ***sometimes*** be much more efficient than D&C
- Greedy algorithms are usually
  - Very fast, but hard to prove optimality for
  - Structured as follows…

The Greedy Method

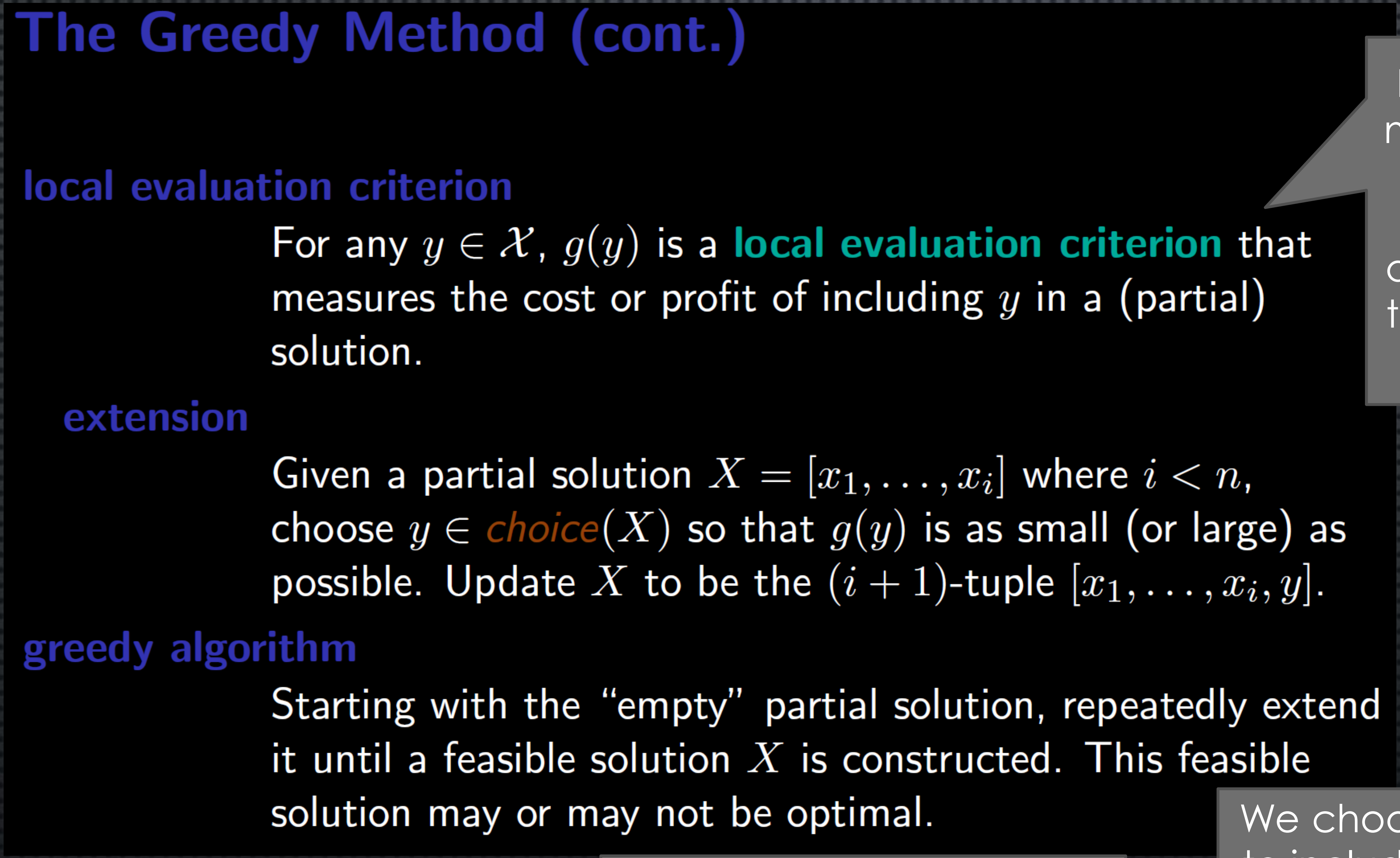
partial solutions

Given a problem instance I, it should be possible to write a feasible solution X as a tuple $[x_1, x_2, \ldots, x_n]$ for some integer n, where $x_i \in \mathcal{X}$ for all i. A tuple $[x_1, \ldots, x_i]$ where $i < n$ is a **partial solution** if no constraints are violated. Note: it may be the case that a partial solution cannot be extended to a feasible solution.

choice set

For a partial solution $X = [x_1, \ldots, x_i]$ where $i < n$, we define the **choice set**

$$\mathit{choice}(X) = \{y \in \mathcal{X} : [x_1, \ldots, x_i, y] \text{ is a partial solution}\}.$$



The Greedy Method (cont.)

local evaluation criterion

For any $y \in \mathcal{X}$, $g(y)$ is a **local evaluation criterion** that measures the cost or profit of including y in a (partial) solution.

extension

Given a partial solution $X = [x_1, \ldots, x_i]$ where $i < n$, choose $y \in$ *choice*(X) so that $g(y)$ is as small (or large) as possible. Update X to be the $(i+1)$-tuple $[x_1, \ldots, x_i, y]$.

greedy algorithm

Starting with the "empty" partial solution, repeatedly extend it until a feasible solution X is constructed. This feasible solution may or may not be optimal.

Local evaluation means we **cannot consider future choices** when deciding whether to include y in our solution.

We **irrevocably** decide to include y (or not). We do **not** reconsider.

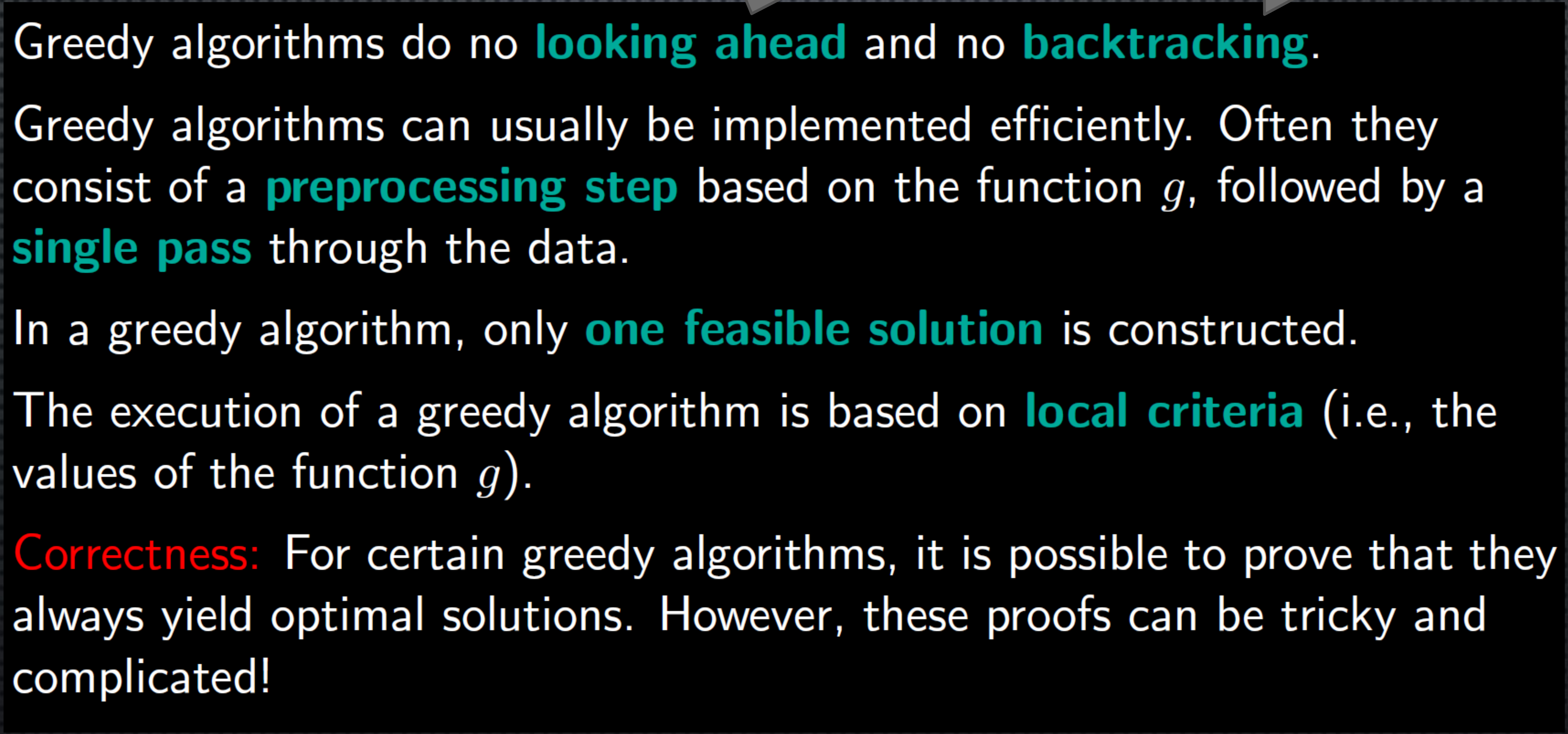
We choose the next element to include **greedily** by taking the y that gives the **maximum local improvement**.

This may or may not be a good idea…

CORE CHARACTERISTICS OF **GREEDY ALGORITHMS**

Cannot consider how your **current** choice affects **future** choices

Cannot undo / change your choice



Greedy algorithms do no **looking ahead** and no **backtracking**.

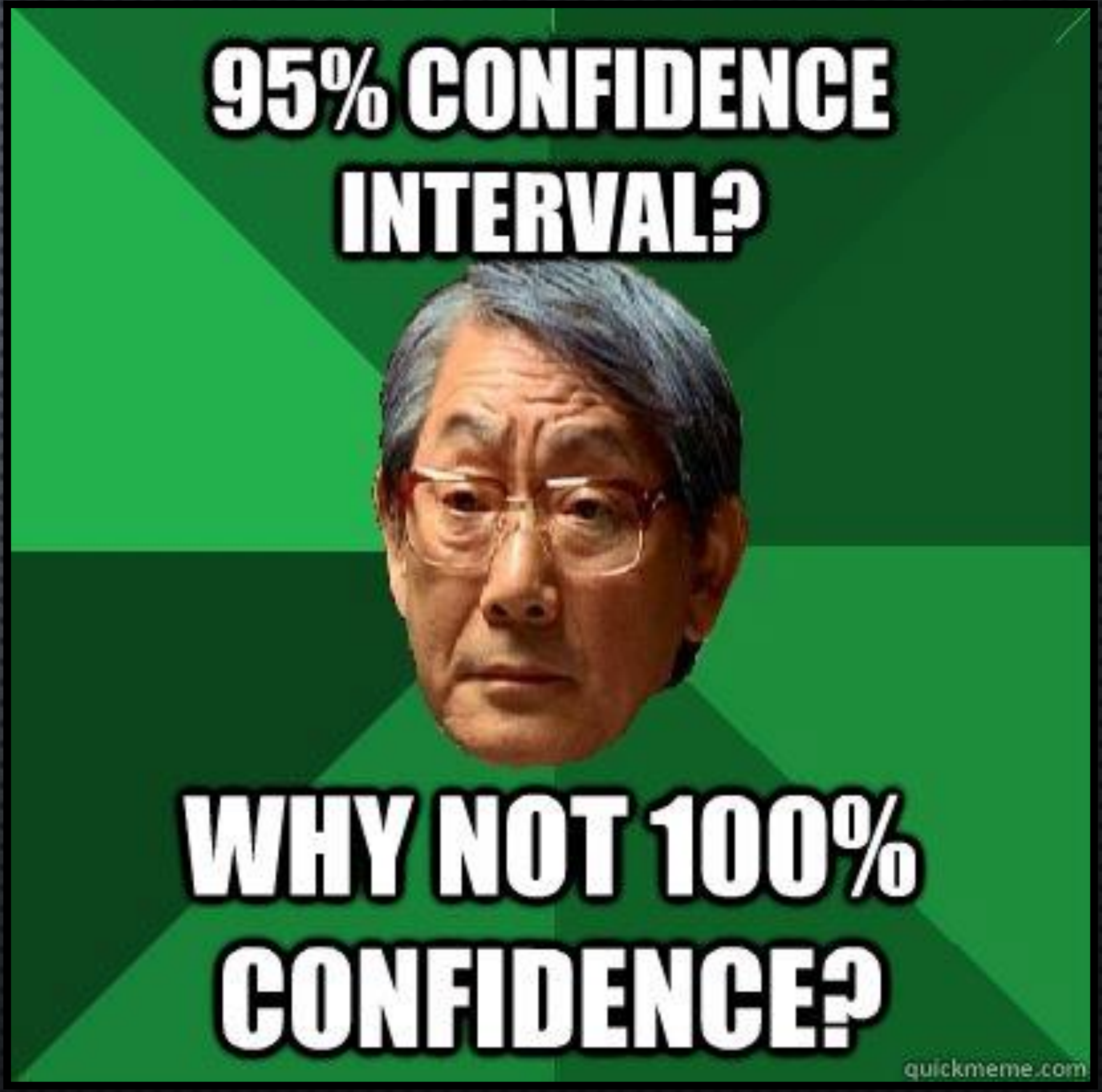
Greedy algorithms can usually be implemented efficiently. Often they consist of a **preprocessing step** based on the function g, followed by a **single pass** through the data.

In a greedy algorithm, only **one feasible solution** is constructed.

The execution of a greedy algorithm is based on **local criteria** (i.e., the values of the function g).

Correctness: For certain greedy algorithms, it is possible to prove that they always yield optimal solutions. However, these proofs can be tricky and complicated!

PROBLEM:
INTERVAL SELECTION



PROBLEM: INTERVAL SELECTION

Where s_i and f_i are positive integers

- **Input:** a set $\boldsymbol{A} = \{\boldsymbol{A_1}, \dots, \boldsymbol{An}\}$ of time intervals
  - Each interval $\boldsymbol{A_i}$ has a start time $\boldsymbol{s_i}$ and a finish time $\boldsymbol{f_i}$
- **Feasible solution:** a subset $\boldsymbol{X}$ of $\boldsymbol{A}$ containing **pairwise disjoint** intervals
- **Output:** a feasible solution of **maximum size**
  - *I.e.,* one that maximizes $|X|$

Bad solution.
Not optimal!

POSSIBLE GREEDY STRATEGIES

1. Sort the intervals in increasing order of starting times. At any stage, choose the **earliest starting** interval that is disjoint from all previously chosen intervals

- **Partial solutions**
  - $X = [x_1, x_2, \ldots, x_i]$ where each x_i is an interval for the output
- **Choices**
  - $\mathcal{X} = A$ (i.e., **all** intervals)
  - $\text{Choice}(X) = \{\, y \in \mathcal{X} : [x_1, \ldots, x_i, y]$ respects all constraints $\}$
    - i.e., where $y \notin X$ and $\forall_{x \in X}\, \text{disjoint}(y, x)$
- **Local evaluation function**
  - $g(y) = s_j$ where $y = A[j]$
  - (i.e., $g(y) =$ start time of interval y)

POSSIBLE GREEDY STRATEGIES FOR INTERVAL SELECTION

1. Sort the intervals in increasing order of starting times. At any stage, choose the **earliest starting** interval that is disjoint from all previously chosen intervals (i.e., the local evaluation criterion is s_i).
2. Sort the intervals in increasing order of duration. At any stage, choose the interval of **minimum duration** that is disjoint from all previously chosen intervals (i.e., the local evaluation criterion is $f_i - s_i$).
3. Sort the intervals in increasing order of finishing times. At any stage, choose the **earliest finishing** interval that is disjoint from all previously chosen intervals (i.e., the local evaluation criterion is f_i).

Does one of these strategies yield a correct greedy algorithm?

STRATEGY 1: PROVING **INCORRECTNESS**

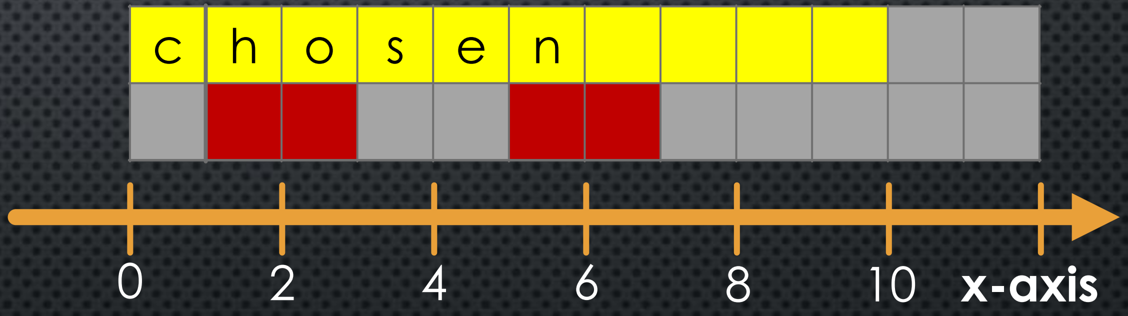
- Idea: find **one input** for which the algorithm gives a **non-optimal** solution **or** an **infeasible** solution

Strategy 1

Sort the intervals in increasing order of starting times. At any stage, choose the earliest starting interval that is disjoint from all previously chosen intervals (i.e., the local evaluation criterion is s_i).

Consider input:

$[0, 10), [1, 3), [5, 7).$



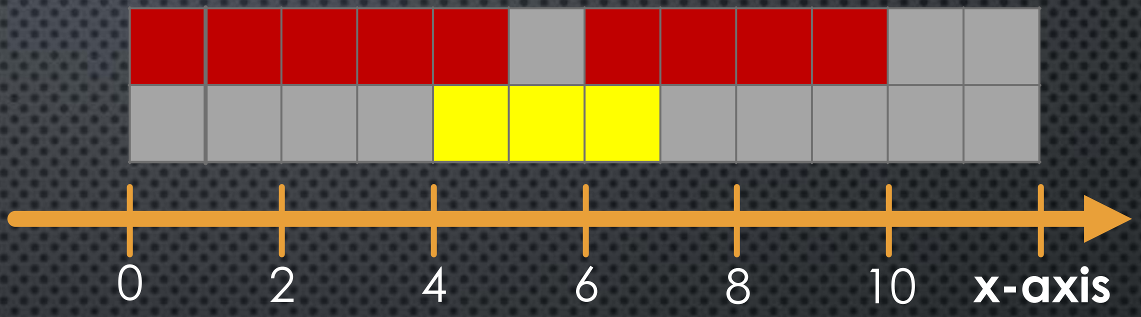
HOW ABOUT STRATEGY 2?

Strategy 2

Sort the intervals in increasing order of duration. At any stage, choose the interval of **minimum duration** that is disjoint from all previously chosen intervals (i.e., the local evaluation criterion is $f_i - s_i$).

Consider input:

$[0, 5), [6, 10), [4, 7)$.



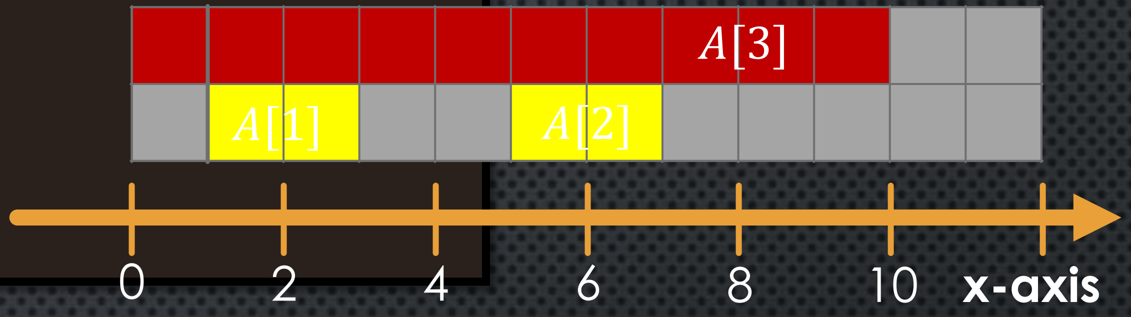
We will show that **Strategy 3** (sort in increasing order of finishing times) always yields the optimal solution.

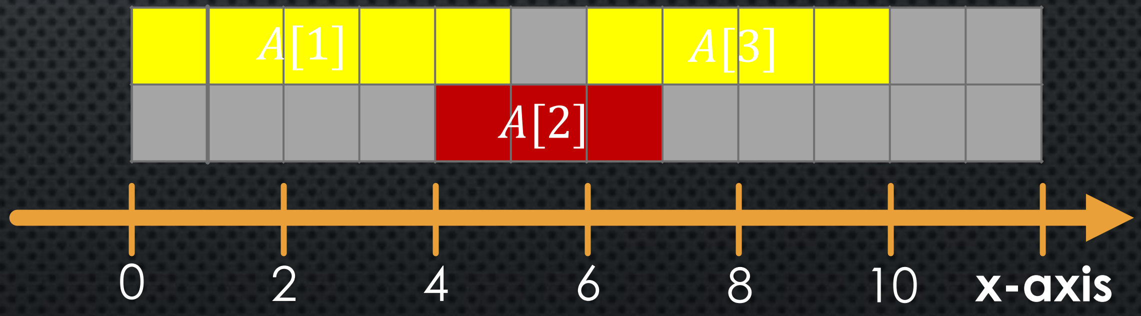
STRATEGY 3

```
GreedyIntervalSelection(A[1..n])
    sort(A) by increasing finish times
    X = [A[1]]
    prev = 1          // index of last selected interval

    for i = 2..n
        if A[i].s >= A[prev].f then
            X.append(A[i])
            prev = i

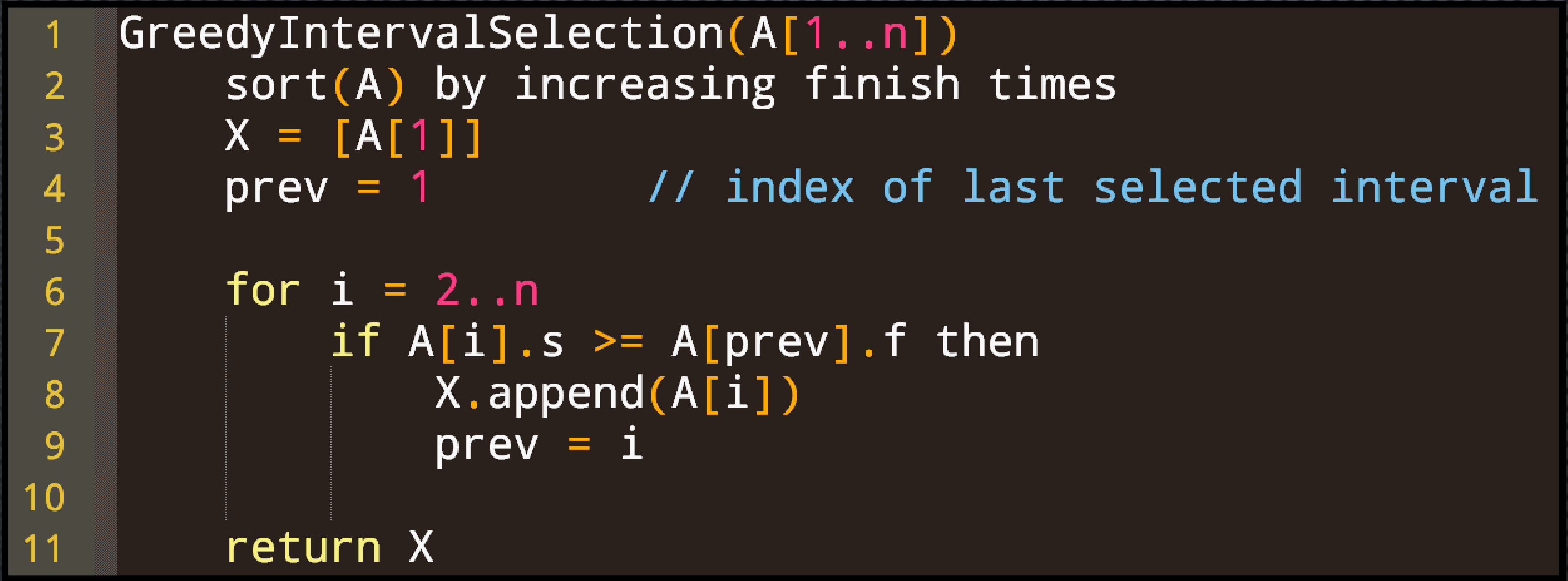
    return X
```





Where is our local evaluation function g in this code?

STRATEGY 3



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    for i = 2..n
        if A[i].s >= A[prev].f then
            X.append(A[i])
            prev = i

    return X
```

Time complexity:

Sort + one pass
$\in \Theta(n \log n)$

How to **prove** this is correct?
(I.e., how can we show the returned solution is both **feasible** and **optimal**?)

Feasibility? Easy!
We always choose an interval that **starts after** all other chosen intervals **end**

Optimality? Harder…


BRO I DON'T WANT PROOF
I WANT EVIDENCE

GREEDY CORRECTNESS PROOFS

- Want to prove: greedy solution X is **correct (feasible & optimal)**
- Usually show **feasibility directly** and **optimality by contradiction:**
  - Suppose solution O is better than X
  - Show this necessarily leads to a contradiction
- Two broad strategies for **deriving** this contradiction:

1. **Greedy stays ahead:** show **every** choice in X is "at least as good" as the corresponding choice in O
2. **Exchange:** show O can be improved by replacing some choice in O with a choice in X

Let's demonstrate approach #1
(next time)