# CS 341: ALGORITHMS

**Lecture 6: greedy algorithms II**

Readings: see website

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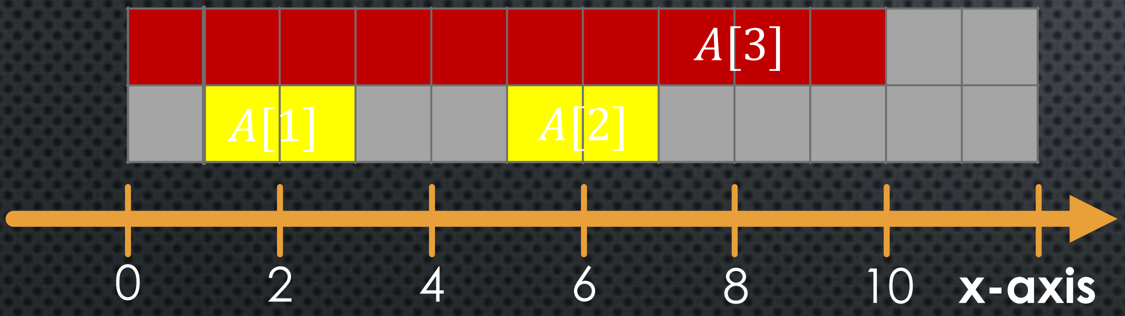
# OPTIMALITY PROOF

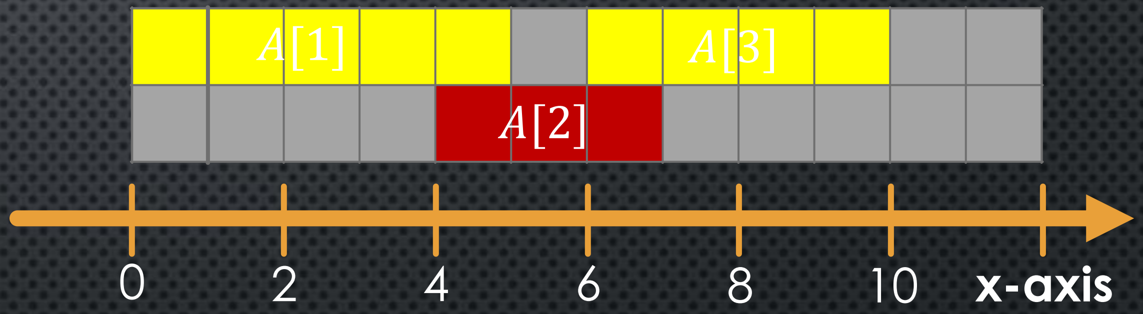
**for greedy interval selection**

**Goal:** choose **as many** disjoint intervals as possible,
(i.e., without any overlap)

**Algorithm:**

3. Sort the intervals in increasing order of finishing times. At any stage, choose the **earliest finishing** interval that is disjoint from all previously chosen intervals (i.e., the local evaluation criterion is $f_i$).

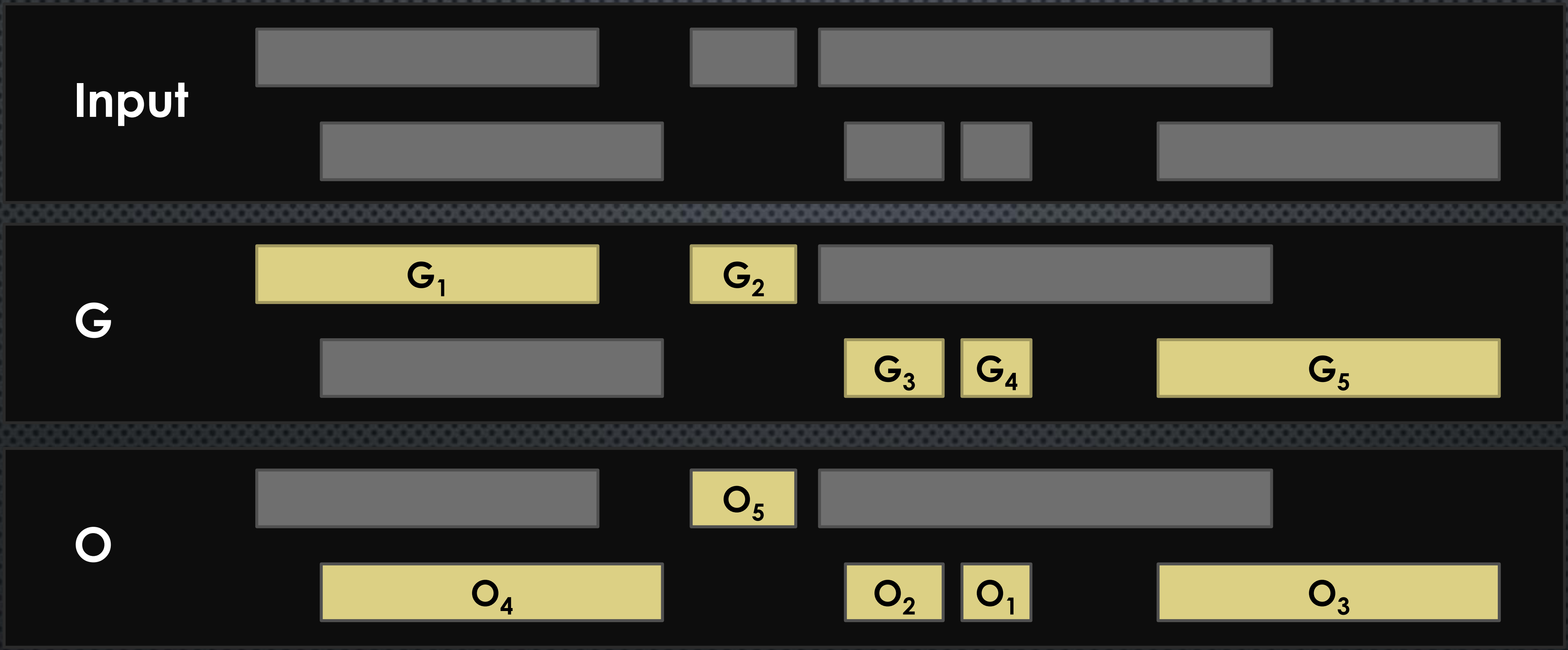


# PROVING OPTIMALITY

- Consider an input A[1..n]
- Let **G** be the greedy solution
- Let **O** be an optimal solution
- “Greedy stays ahead” argument
  - Intuition: out of the a given set of intervals, greedy picks **as many as optimal**

# VISUAL EXAMPLE

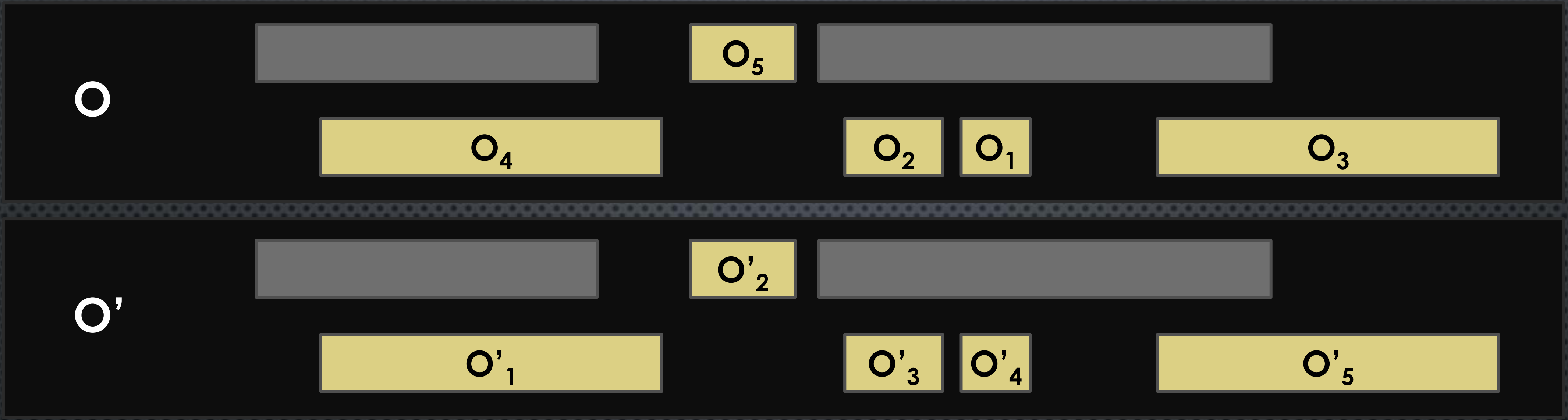



How to compare G and O? **Imagine <u>reordering</u> O to match G!**

**<u>CRUCIAL:</u>** We are **NOT** assuming the optimal **algorithm** uses the same sort order!

We are merely **imagining reordering** the intervals chosen by the optimal algorithm so we can easily **compare their finish times** to intervals in **G**

# REORDERING O BY INCREASING FINISH TIME

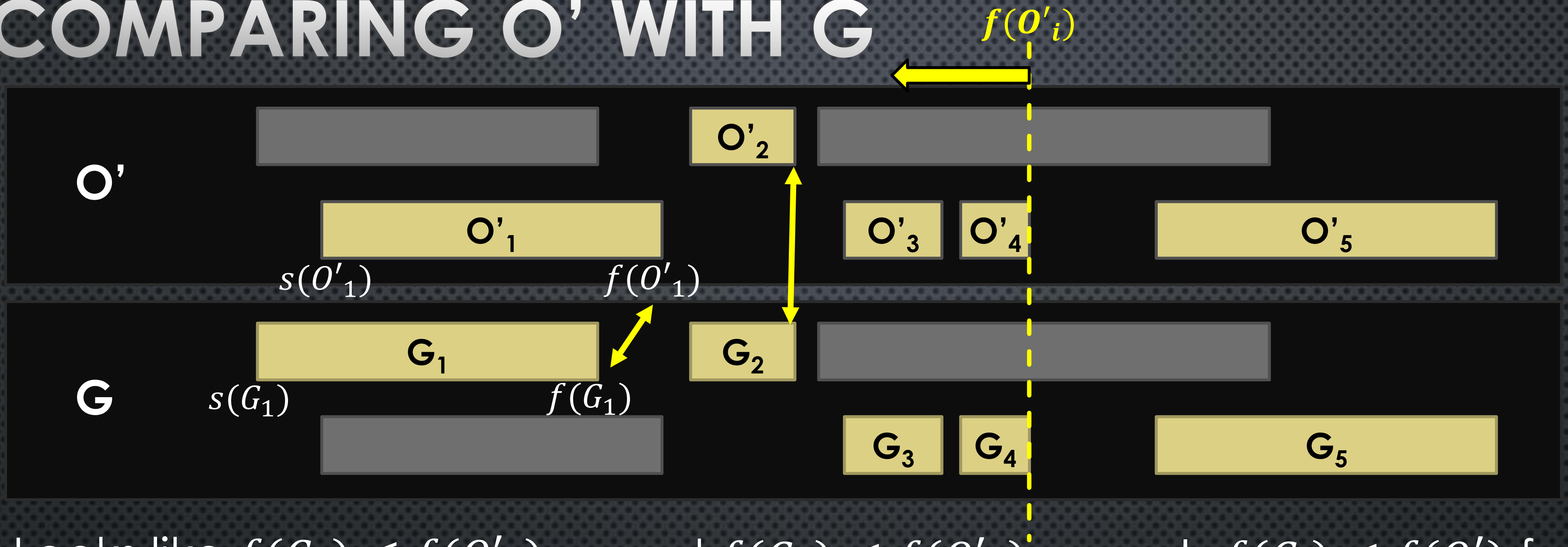



Now O' and G are both ordered by increasing finish time

This ordering helps us leverage what we know about G
in our comparison with O'.

Argue for a prefix of the intervals sorted this way, G chooses **as many as O'**

# COMPARING O' WITH G



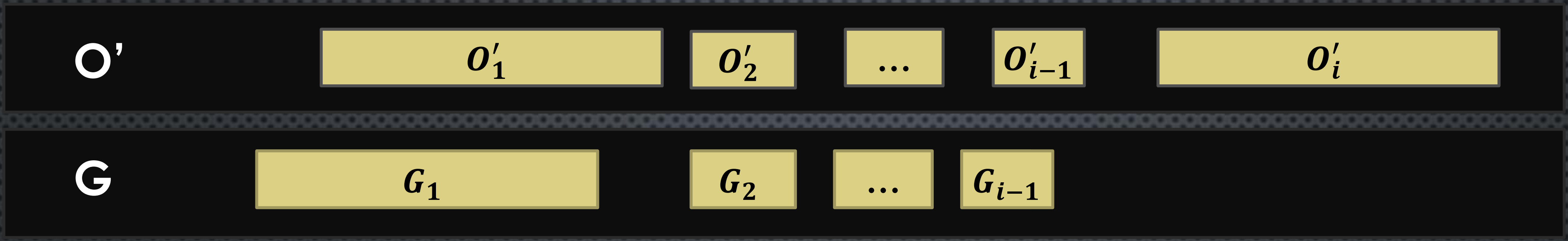

Looks like $f(G_1) \le f(O'_1)$ and $f(G_2) \le f(O'_2)$ … Is $f(G_i) \le f(O'_i)$ for **all** $i$?

**If** this trend holds in general, then

**out of the intervals with finish time** $\le f(O'_i)$

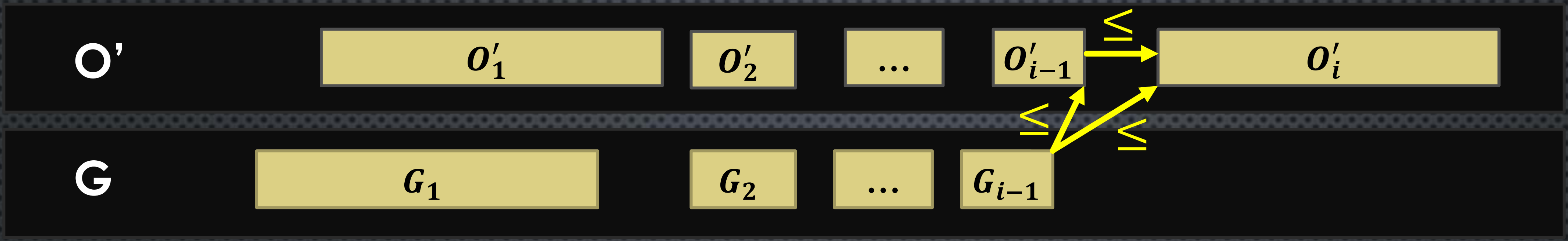
**G** chooses **as many** intervals as **O**!

# PROVING LEMMA: $f(G_i) \leq f(O'_i)$ FOR ALL $i$




Base case: $f(G_1) \leq f(O'_1)$ since **G** chooses the interval with the earliest finish time first.

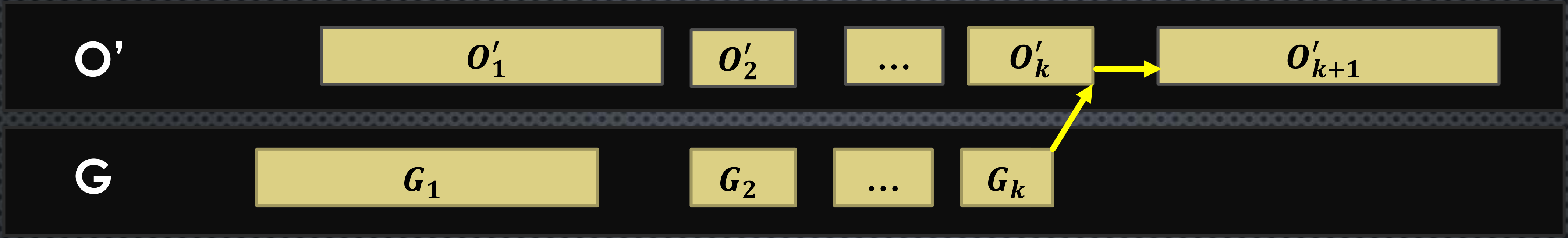
# PROVING **LEMMA**: $f(G_i) \leq f(O'_i)$ FOR **ALL** $i$




Inductive step: assume $f(G_{i-1}) \leq f(O'_{i-1})$. Show $f(G_i) \leq f(O'_i)$.

- Since O' is feasible, $f(O'_{i-1}) \leq s(O'_i)$
- So $f(G_{i-1}) \leq s(O'_i)$
- So $G$ **can choose** $O'_i$ if it has the smallest finish time
- **So $f(G_i) \leq f(O'_i)$**

# USING THIS LEMMA




- Suppose $|O'| > |G|$ to obtain a contradiction
  - So if $G$ chooses $k$ intervals, $O'$ chooses at least $k+1$
- By the lemma, $f(G_k) \leq f(O_k)$
- Since $O'$ is feasible, $f(O'_k) \leq s(O'_{k+1})$
- **But then $G$ can, and would, pick $O'_{k+1}$.**
  - **Contradiction!**

So assumption $|O'| > |G|$ is wrong!

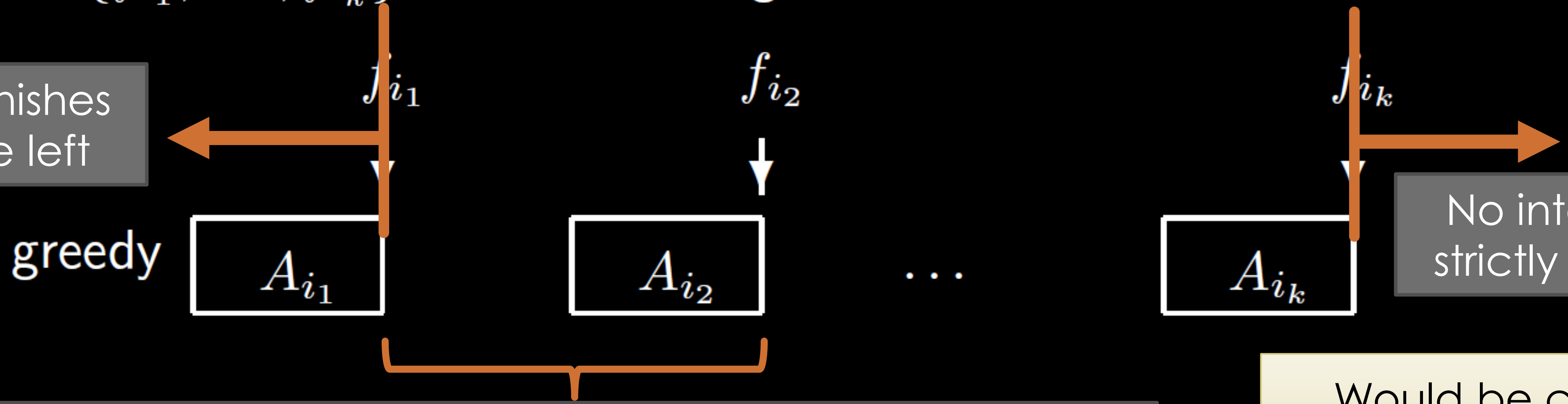
So $\boldsymbol{G}$ is optimal

# A DIFFERENT PROOF

**"Slick"** ad-hoc approaches are sometimes possible…

Let $F = \{f_{i_1}, \ldots, f_{i_k}\}$ be the finishing times of the intervals in $X$

No interval finishes strictly to the left



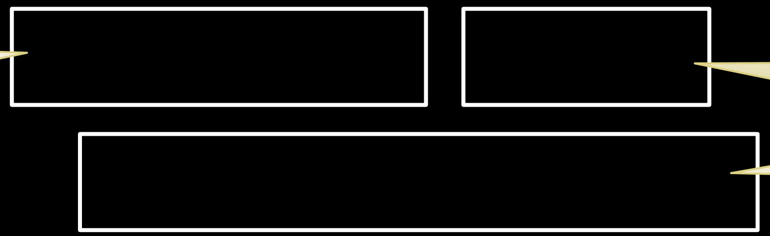

No interval starts strictly to the right

**No** interval in is strictly between these points!

Would be chosen by greedy! (contradiction)

So, in addition to the intervals in $X$, only the following types of intervals are **possible**

Contains $f_{i_1}$



Contains $f_{i_2}$

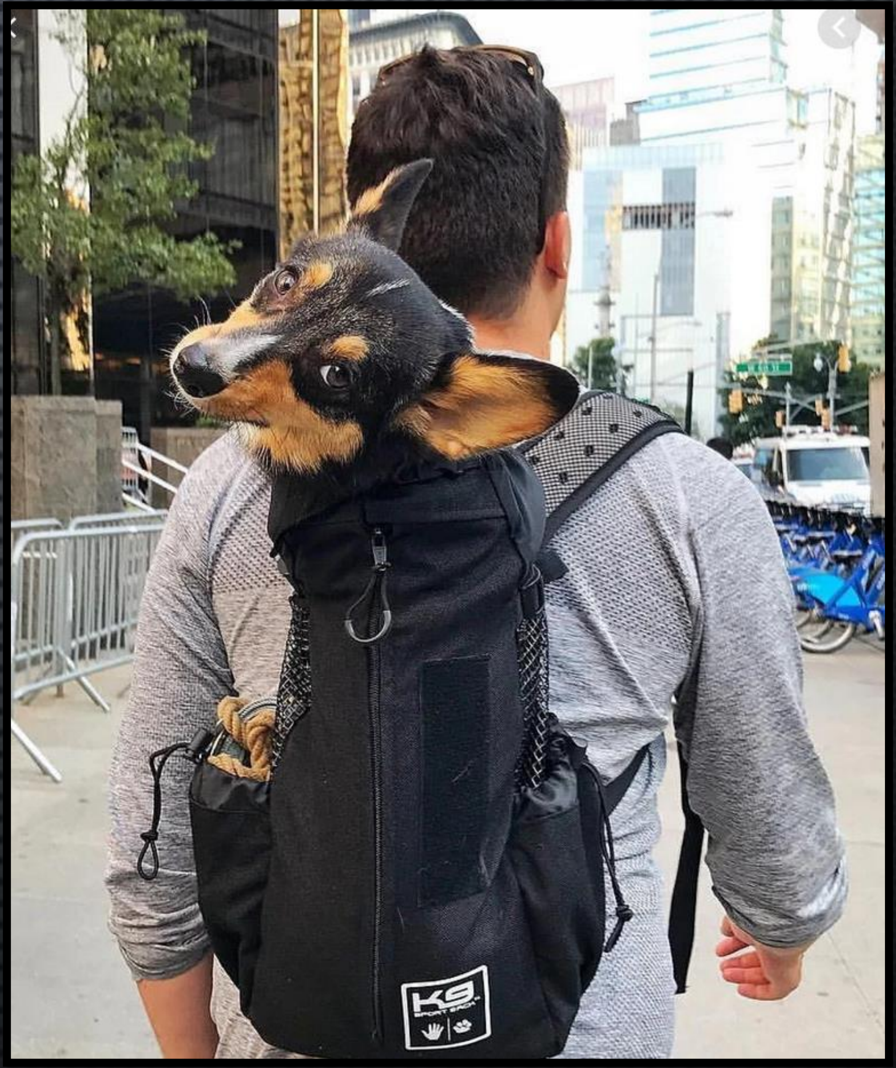
Contains $f_{i_1}$ **and** $f_{i_2}$

Thus, **every interval** contains some finishing time in $F$

And, two intervals in $O$ **cannot contain the same** element of $F$

So, there must be as many finishing times in $F$ as there are intervals in $O$. QED

# KNAPSACK PROBLEMS



## Problem 4.4

**Knapsack**

**Instance:** **Profits** $P = [p_1, \ldots, p_n]$; **weights** $W = [w_1, \ldots, w_n]$; *and a* **capacity**, $M$. *These are all positive integers.*

**Feasible solution:** *An* $n$*-tuple* $X = [x_1, \ldots, x_n]$ *where* $\sum_{i=1}^{n} w_i x_i \leq M$.

Gotta respect the **weight limit M…**

### Problem 4.4

**Knapsack**

**Instance:** **Profits** $P = [p_1, \ldots, p_n]$; **weights** $W = [w_1, \ldots, w_n]$; *and a* **capacity**, $M$. *These are all positive integers.*

**Feasible solution:** *An* $n$*-tuple* $X = [x_1, \ldots, x_n]$ *where* $\sum_{i=1}^{n} w_i x_i \leq M$.

*In the* **0-1 Knapsack** *problem (often denoted just as* **Knapsack***), we require that* $x_i \in \{0, 1\}$, $1 \leq i \leq n$.

*In the* **Rational Knapsack** *problem, we require that* $x_i \in \mathbb{Q}$ *and* $0 \leq x_i \leq 1$, $1 \leq i \leq n$.

**Find:** *A feasible solution* $X$ *that maximizes* $\sum_{i=1}^{n} p_i x_i$.

**0-1 Knapsack:**
NP Hard.
Probably requires exponential time to solve...

**Rational knapsack:**
Can be solved in polynomial time by a greedy alg!

**Lets discuss this now… other one later**

# POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 1:** consider items in **decreasing** order of **profit** (i.e., we maximize the local evaluation criterion $\boldsymbol{p_i}$)
- Let's try an example input
  - Profits $P = [20, 50, \mathbf{100}]$
  - Weights $W = [10, 20, 10]$
  - Weight limit $M = 10$
- Algorithm selects last item for 100 profit
  - Looks optimal in this example

# POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 1:** consider items in **decreasing** order of **profit** (i.e., we maximize the local evaluation criterion $\boldsymbol{p_i}$)
- How about a **second example input**
  - Profits $P = [20,50,\mathbf{100}]$
  - Weights $W = [10,20,100]$
  - Weight limit $M = 10$
- Algorithm selects last item for **10** profit
  - **Not optimal!**

# POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 2:** consider items in **increasing** order of **weight** (i.e., we minimize the local evaluation criterion $\boldsymbol{w_i}$)
- **Counterexample**
  - Profits $P = [20{,}50{,}100]$
  - Weights $W = [\mathbf{10}, 20{,}100]$
  - Weight limit $M = 10$
- Algorithm selects first item for 20 profit
  - It **could** select half of second item, for 25 profit!

# POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 3:** consider items in **decreasing** order of **profit divided by weight** (i.e., we maximize local evaluation criterion $\boldsymbol{p_i/w_i}$)
- Let's try our first example input
  - Profits $P = [20,50,\mathbf{100}]$
  - Weights $W = [10,20,10]$
  - Weight limit $M = 10$
- Profit divided by weight
  - $P/W = [2, 2.5, 10]$
- Algorithm selects last item for 100 profit (optimal)

# POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 3:** consider items in **decreasing** order of **profit divided by weight** (i.e., we maximize local evaluation criterion $\boldsymbol{p_i/w_i}$)
- Let's try our second example input
  - Profits $P = [20,50,\mathbf{100}]$
  - Weights $W = [10,20,100]$
  - Weight limit $M = 10$
- Profit divided by weight
  - $P/W = [2, 2.5, 1]$
- Algorithm selects second item for 25 profit (optimal)

It turns out strategy #3 **is** optimal…

```
Preprocess(A[1..n], M) // A[i] = (p_i, w_i)
    sort A by decreasing profit divided by weight
    let p[1..n] be the profits in A
    let w[1..n] be the weights in A
    return GreedyRationalKnapsack(p, w, M)

GreedyRationalKnapsack(p[1..n], w[1..n], M)
    X = [0, ..., 0]
    weight = 0

    for i = 1..n
        if weight + w[i] > M then
            X[i] = (M - weight) / w[i]
            break
        else
            X[i] = 1
            weight = weight + w[i]

    return X
```

No items are chosen yet

Current weight of knapsack

For all items

If we **cannot** fit the entire item

Put in as much of the item as you can, to **exactly fill** the knapsack

Otherwise take the entire item

Either X=(1,1,…,1,0,…,0) or X=(1,1,…,1,$x_i$,0,…,0) where $x_i \in (0,1)$

```
Preprocess(A[1..n], M) // A[i] = (p_i, w_i)
    sort A by decreasing profit divided by weight
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    for i = 1..n
        if weight + w[i] > M then
            X[i] = (M - weight) / w[i]
            break
        else
            X[i] = 1
            weight = weight + w[i]

    return X
```

Running time complexity?

Can do preprocessing in $\Theta(n \log n)$

Create array in $\Theta(n)$ time

$\Theta(n)$ iterations each doing $\Theta(1)$ work

Total $\boldsymbol{\Theta(n \log n)}$ (or $\Theta(n)$ if input is already sorted)

# **INFORMAL** FEASIBILITY ARGUMENT

## (SHOULD BE GOOD ENOUGH TO SHOW FEASIBILITY ON ASSESSMENTS)

- **Feasibility: all $x_i$ are in $[0, 1]$ and total weight is $\leq M$**
- Either *everything* fits in the knapsack, or:
- When we exit the loop, **weight is exactly M**
- Every time we write to $x_i$ it's either 0, 1 or $(M - weight)/w_i$ where $weight + w[i] > M$
  - Rearranging the latter we get $(M - weight)/w_i < 1$
  - And $weight \leq M$, so $(M - weight)/w_i \geq 0$
  - **So, we have $x_i \in [0, 1]$**

```
for i = 1..n
    if weight + w[i] > M then
        X[i] = (M - weight) / w[i]
        break
    else
        X[i] = 1
        weight = weight + w[i]
```

# MINOR MODIFICATION TO FACILITATE **FORMAL** PROOF

```
GreedyRationalKnapsack(p[1..n], w[1..n], M)
    X = [0, ..., 0]
    weight = 0

    for i = 1..n
        if weight + w[i] > M then
            X[i] = (M - weight) / w[i]
            weight = M
            break
        else
            X[i] = 1
            weight = weight + w[i]

    return X
```

Does NOT change behaviour of the algorithm at all!

**Optional slide, just for your notes**

# FORMAL FEASIBILITY ARG

```
for i = 1..n
    if weight + w[i] > M then
        X[i] = (M - weight) / w[i]
        weight = M
        break
    else
        X[i] = 1
        weight = weight + w[i]
```

Optional slide, just for your notes

- **Loop invariant:** $\forall_i : x_i \in [0,1]$
- **and** $weight = \sum_{i=1}^{n} w_i x_i \leq M$
  - Base case. Initially weight = 0 and $\forall_i : x_i = 0$.
    - So $0 = weight = \sum_{i=1}^{n} w_i \cdot 0 = \sum_{i=1}^{n} w_i x_i \leq M$
  - Inductive step.
    - Suppose invariant holds at start of iteration $i$
    - Let $weight', x_i'$ denote values of $weight, x_i$ at **end** of iteration $i$
    - Prove invariant holds at end of iteration $i$
    - i.e., $\forall_i : x_i' \in [0,1]$ **and** $weight' = \sum_{i=1}^{n} w_i x'_i \leq M$

# FORMAL FEASIBILITY ARG

```
5         for i = 1..n
6             if weight + w[i] > M then
7                 X[i] = (M - weight) / w[i]
8                 weight = M
9                 break
10            else
11                X[i] = 1
12                weight = weight + w[i]
```

**Optional slide, just for your notes**

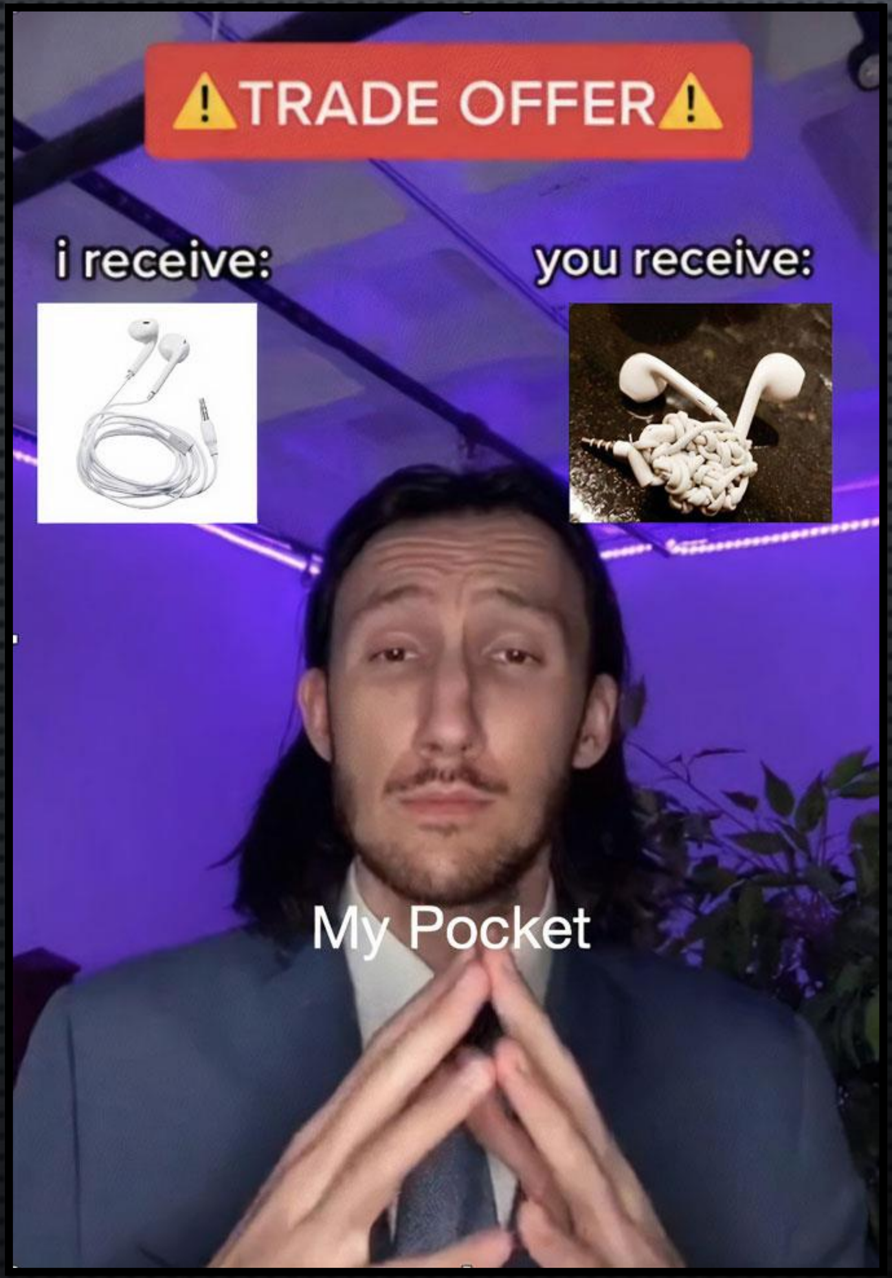
- **WTP:** $\forall_i : x_i' \in [0,1]$
  **and** $weight' = \sum_{i=1}^{n} w_i x'_i \leq M$
- Case 1: $weight + w_i \leq M$
  - $x_i' = 1$ **which is in** $[0,1]$ (by line 11)
  - $weight' = weight + w_i$ (by line 12)
    and **this is** $\leq M$ by the case
  - $weight' = \sum_{k=1}^{n} x_k w_k + w_i$ (by invariant)
  - $weight' = \sum_{k=1}^{n} x_k w_k + x_i' w_i$ (since $x_i' = 1$)
  - And $x_k' = x_k$ for all $k \neq i$ and $x_i = 0$ so $\sum_{k=1}^{n} x_k' w_k = x_i' w_i + \sum_{k=1}^{n} x_k w_k$
  - Rearrange to get $\sum_{k=1}^{n} x_k w_k = (\sum_{k=1}^{n} x_k' w_k - x_i' w_i)$
  - So $weight' = (\sum_{k=1}^{n} x_k' w_k - x_i' w_i) + x_i' w_i = \sum_{k=1}^{n} x_k' w_k$

# FORMAL FEASIBILITY ARG

```
5       for i = 1..n
6           if weight + w[i] > M then
7               X[i] = (M - weight) / w[i]
8               weight = M
9               break
10          else
11              X[i] = 1
12              weight = weight + w[i]
```

Optional slide, just for your notes

- **WTP:** $\forall_i : x'_i \in [0,1]$ **and $weight' = \sum_{i=1}^{n} w_i x'_i \leq M$**
- Case 2: $weight + w_i > M$
  - We have $w_i > M - weight$ (by case) and $M - weight \geq 0$ (by invariant)
  - So $0 \leq \frac{M - weight}{w_i} < 1$ **which means $x'_i \in [0, 1)$**
  - $\boldsymbol{weight' = M} = weight + (M - weight)$ (by line 8)
  - $weight' = \sum_{k=1}^{n} x_k w_k + (M - weight)$ (by invariant)
  - But $x'_k = x_k$ for all $k \neq i$ and $x_i = 0$ so $\sum_{k=1}^{n} x'_k w_k = x'_i w_i + \sum_{k=1}^{n} x_k w_k$
  - Rearrange to get $\sum_{k=1}^{n} x_k w_k = \left(\sum_{k=1}^{n} x'_k w_k - x'_i w_i\right)$
  - So $weight' = \left(\sum_{k=1}^{n} x'_k w_k - x'_i w_i\right) + (M - weight)$
  - And $M - weight = x'_i w_i$ **so $weight' = \sum_{k=1}^{n} x'_k w_k$**




# EXCHANGE ARGUMENT

for proving optimality

# OPTIMALITY – AN **EXCHANGE** ARUGMENT

For simplicity, assume that the profit / weight ratios are all distinct, so

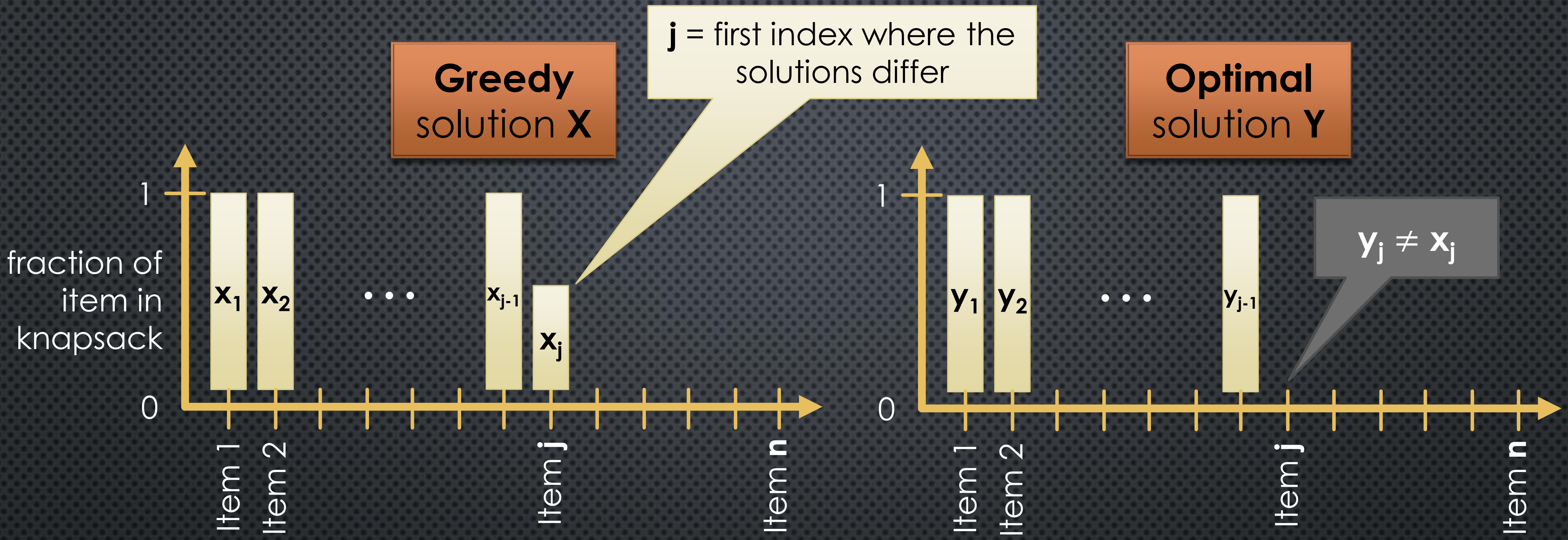
$$\frac{p_1}{w_1} > \frac{p_2}{w_2} > \cdots > \frac{p_n}{w_n}.$$

Suppose the greedy solution is $X = (x_1, \ldots, x_n)$ and the optimal solution is $Y = (y_1, \ldots, y_n)$.

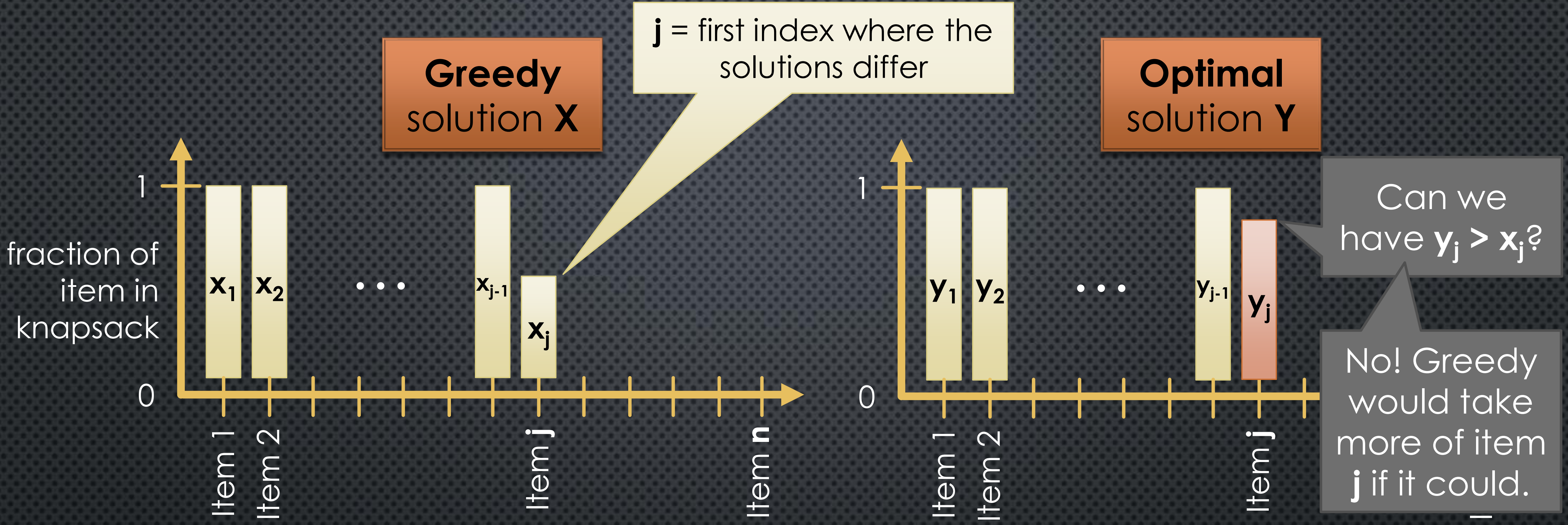
We will prove that $X = Y$, i.e., $x_j = y_j$ for $j = 1, \ldots, n$. Therefore there is a unique optimal solution and it is equal to the greedy solution.

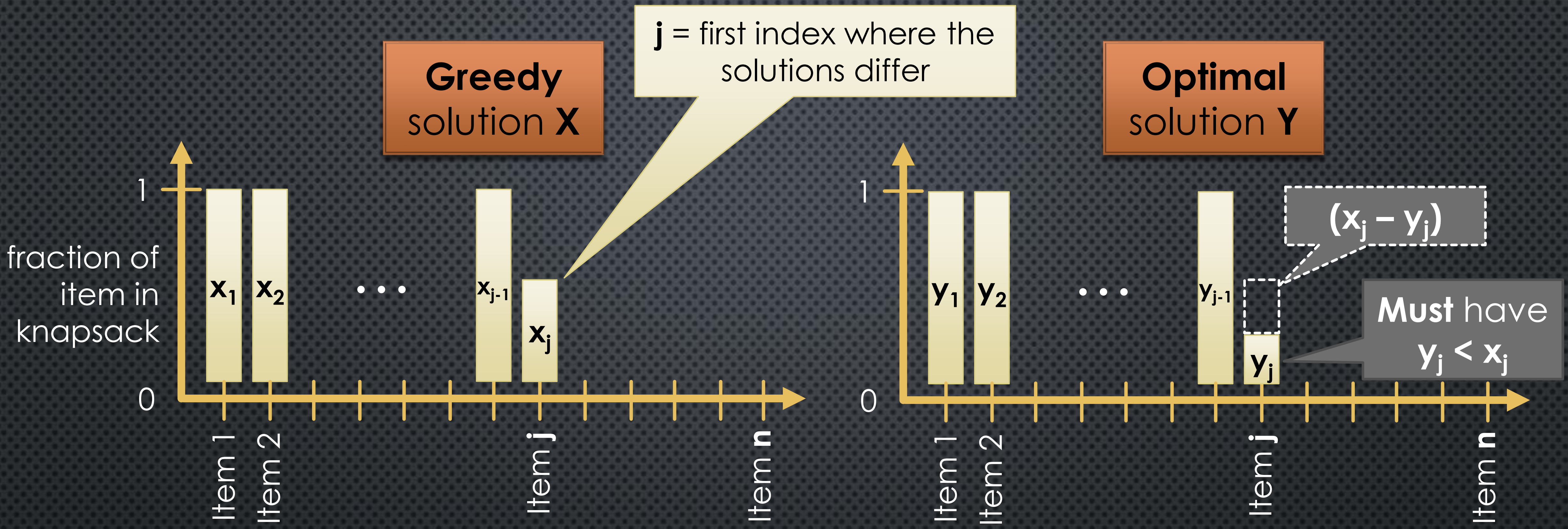
Suppose $X \neq Y$. — To obtain a contradiction

Pick the smallest integer $j$ such that $x_j \neq y_j$. — $X$ and $Y$ are **identical** up to $x_j$ and $y_j$, respectively

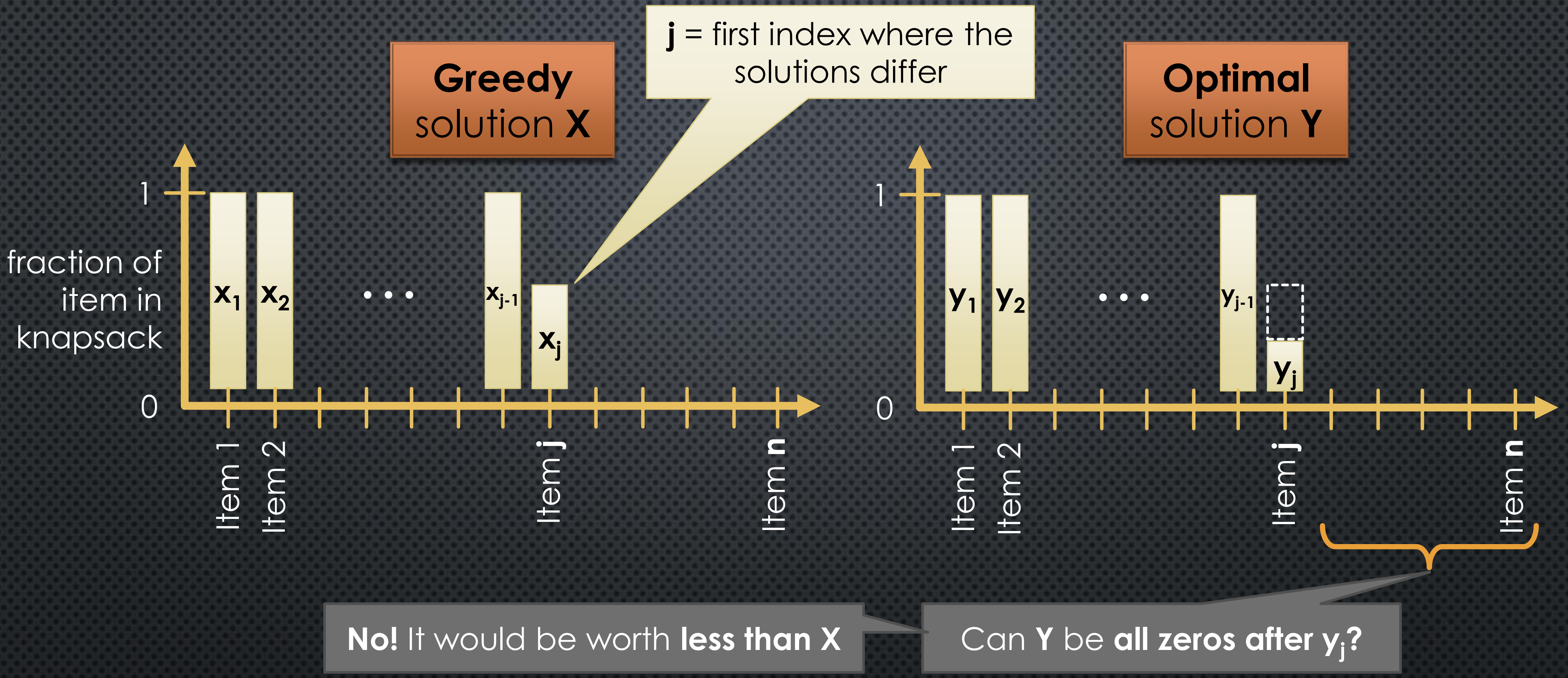



What's the relationship between $x_j$ and $y_j$?


Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
$x_1$
$x_2$
$x_{j-1}$
$x_j$
Item 1
Item 2
Item j
Item n
$y_1$
$y_2$
$y_{j-1}$
$y_j$
Can we have $y_j > x_j$?
No! Greedy would take more of item j if it could.

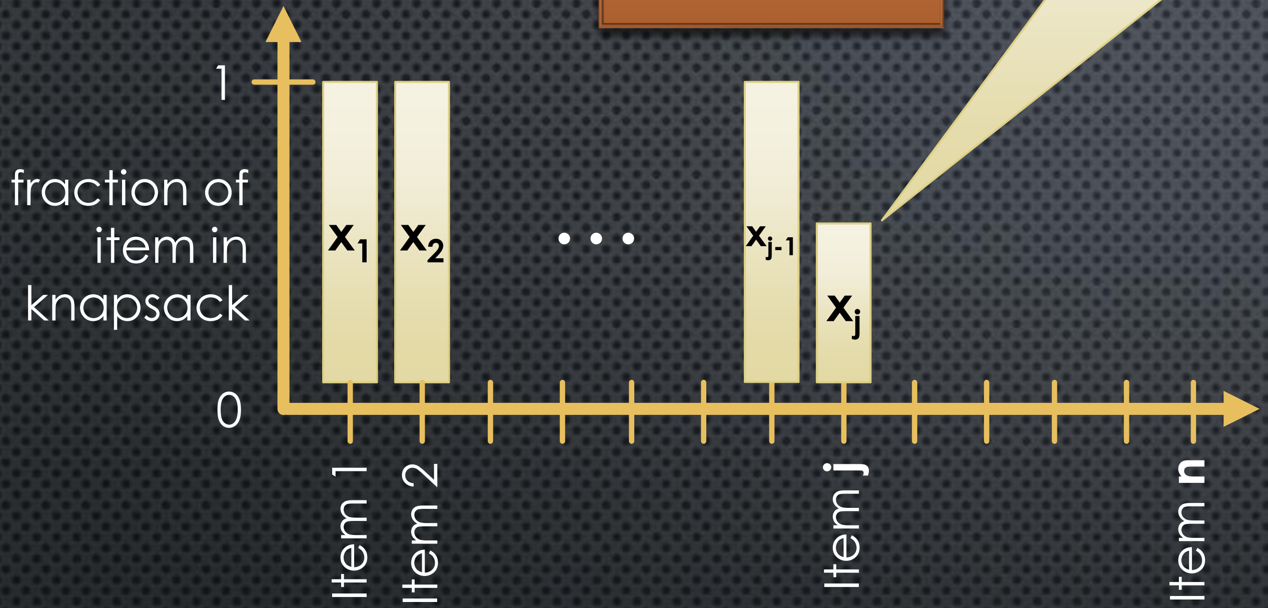


Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
$x_1$
$x_2$
• • •
$x_{j-1}$
$x_j$
Item 1
Item 2
Item j
Item n
1
0
$y_1$
$y_2$
• • •
$y_{j-1}$
$y_j$
$(x_j – y_j)$
Must have $y_j < x_j$
Item 1
Item 2
Item j
Item n



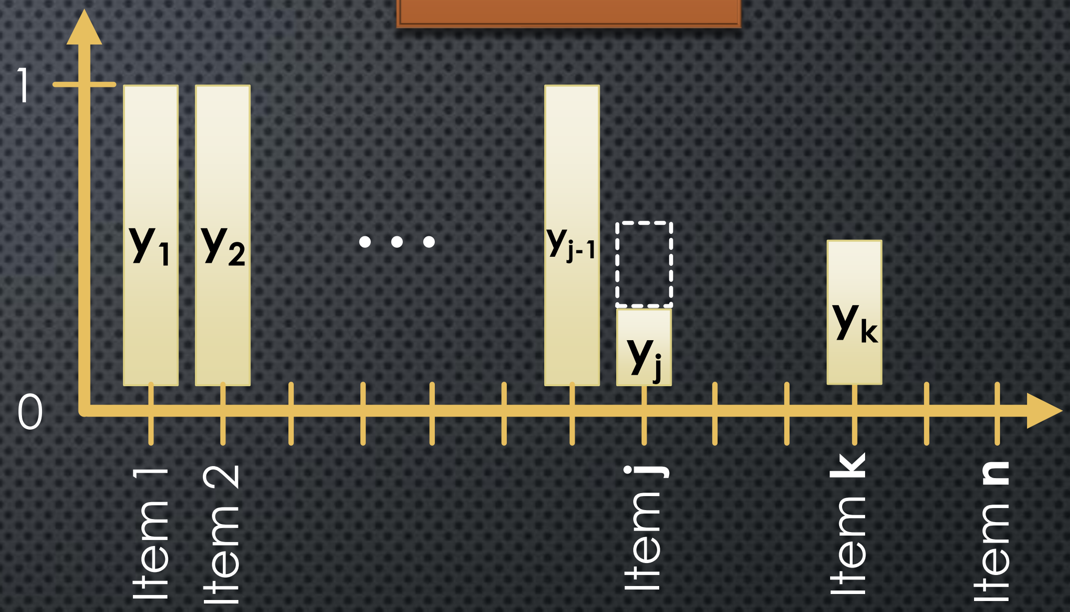
Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
x1
x2
xj-1
xj
Item 1
Item 2
Item j
Item n
y1
y2
yj-1
yj
Item 1
Item 2
Item j
Item n
Can Y be all zeros after yj?
No! It would be worth less than X

**Greedy** solution **X**




**j** = first index where the solutions differ

**Optimal** solution **Y**

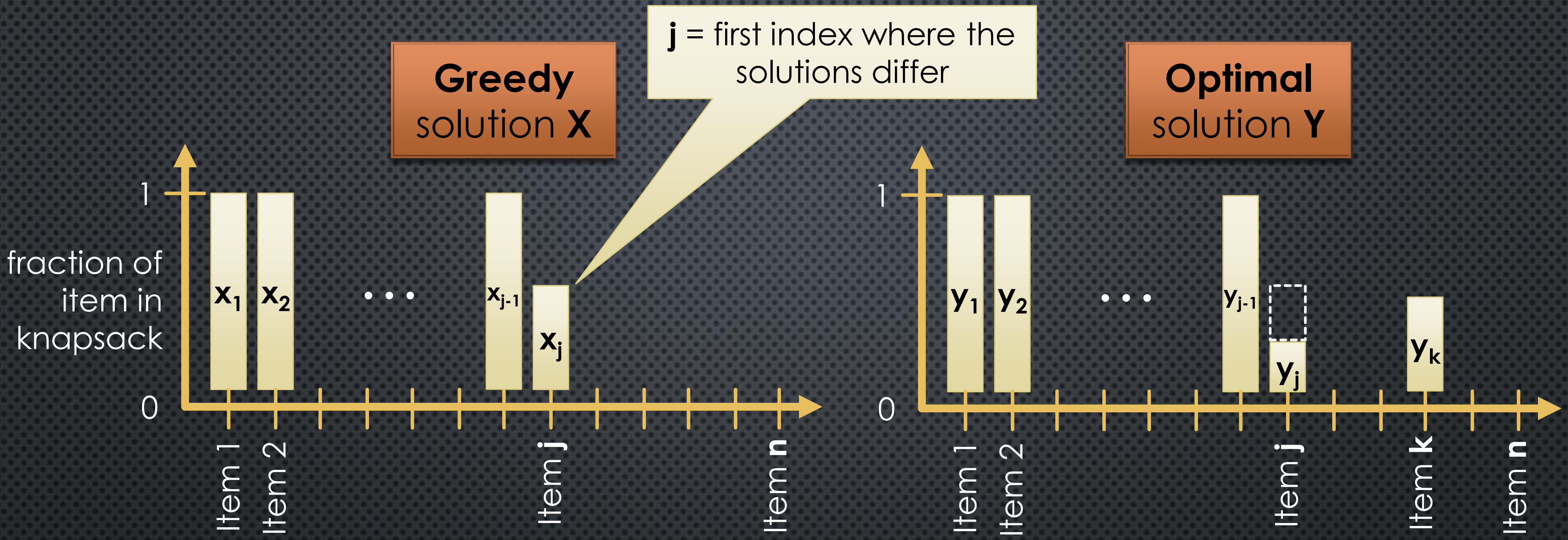



Must exist **k > j** such that **$y_k$ > 0**

But, by our **sort order**, **item j is worth more** (per unit of weight) than **item k**!

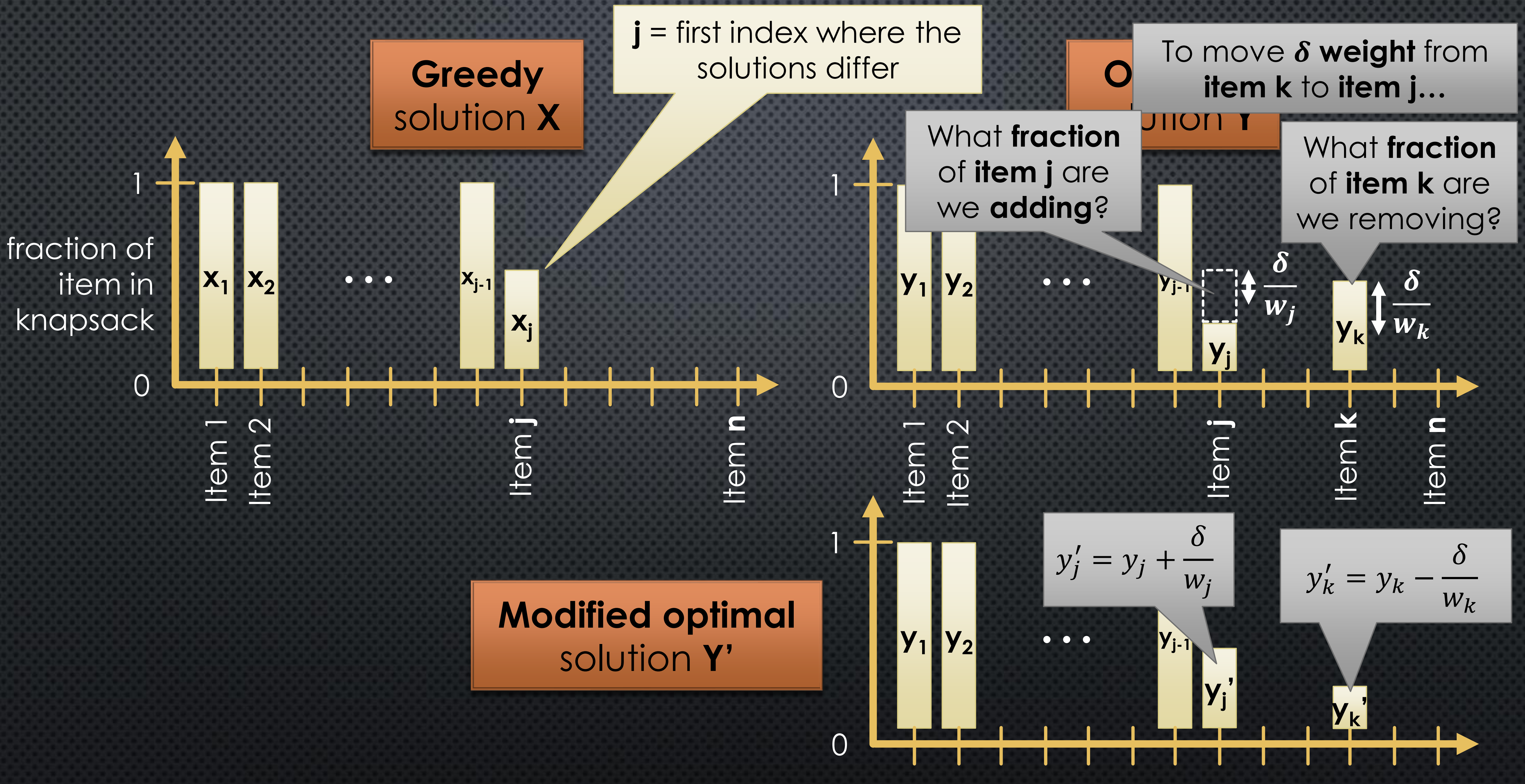
**Remove** some of item **k** and **replace** it with some of item **j**?

**How much** of item k should we remove?



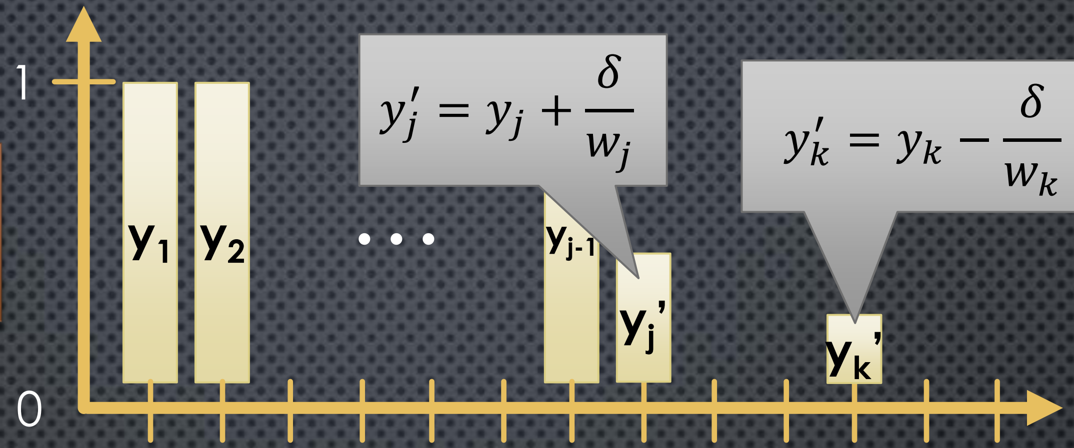

Since item j is worth **more per unit weight**, replacing **even a tiny amount** of item k with item j will improve the solution

So, we remove an infinitesimal $\delta > 0$ of weight of item k, and add $\delta$ weight of item j



Greedy solution X
j = first index where the solutions differ
fraction of item in knapsack
1
0
$x_1$ $x_2$ … $x_{j-1}$ $x_j$
Item 1
Item 2
Item j
Item n
To move $\delta$ weight from item k to item j…
Optimal solution Y
What fraction of item j are we adding?
What fraction of item k are we removing?
$\frac{\delta}{w_j}$
$\frac{\delta}{w_k}$
$y_1$ $y_2$ … $y_{j-1}$ $y_j$ $y_k$
Item k
Modified optimal solution Y'
$y_j' = y_j + \frac{\delta}{w_j}$
$y_k' = y_k - \frac{\delta}{w_k}$
$y_j'$ $y_k'$

**Modified optimal** solution **Y'**




The idea is to show that

$Y'$ is feasible, and

profit$(Y')$ > profit$(Y)$.

This contradicts the optimality of $Y$ and proves that $X = Y$.

To show $Y'$ is feasible, we show $y'_k \geq 0, y'_j \leq 1$ and $weight(Y') \leq M$

# FEASIBILITY OF $Y'$

- To show $Y'$ is feasible, we show $y'_k \geq 0, y'_j \leq 1$ and $weight(Y') \leq M$
- Let's show $y'_k \geq 0$
  - By definition, $y'_k = y_k - \frac{\delta}{w_k}$
  - So, $y'_k \geq 0$ iff $y_k - \frac{\delta}{w_k} \geq 0$ iff $\delta \leq y_k w_k$
  - And we know $y_k$ and $w_k$ are both **positive**
  - So, this constrains $\delta$ to be smaller than this **positive number**
  - Therefore, it is possible to choose positive $\delta$ s.t. $y'_k \geq 0$

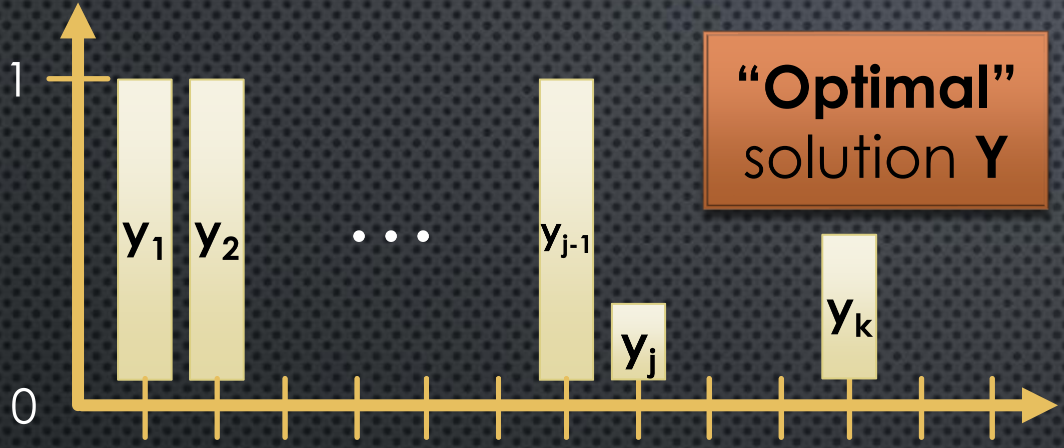
**Existence proof**, but a **non-constructive** one

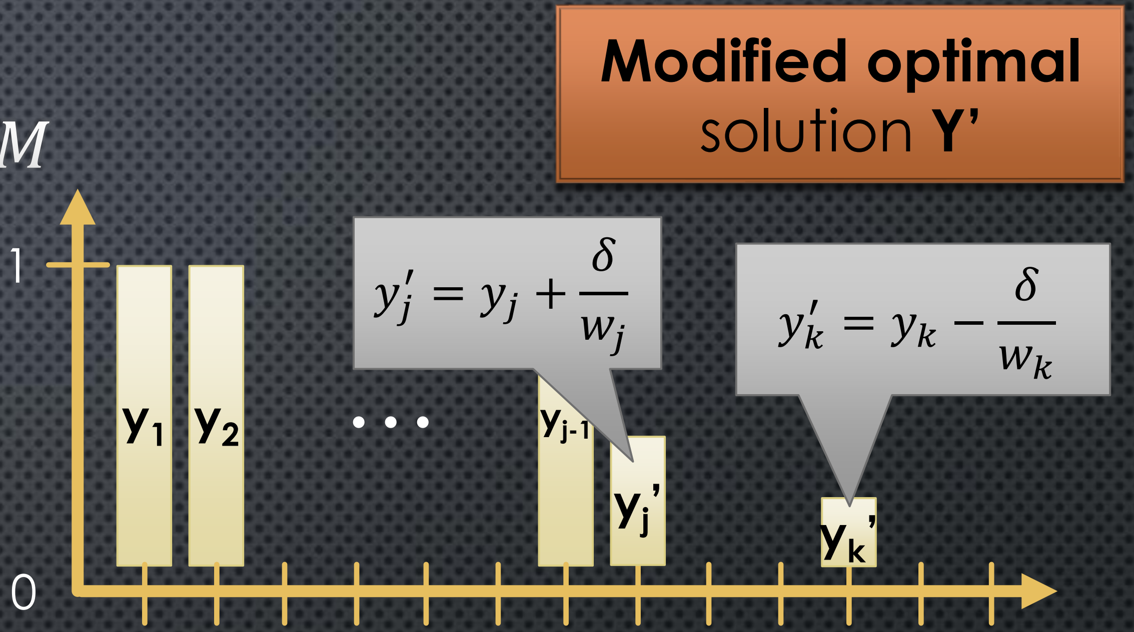
# FEASIBILITY OF $Y'$

- To show $Y'$ is feasible, we show $y_k' \geq 0, y_j' \leq 1$ and $weight(Y') \leq M$
- Now let's show $y_j' \leq 1$
  - By definition, $y_j' = y_j + \frac{\delta}{w_j}$
  - So, $y_j' \leq 1$ iff $y_j + \frac{\delta}{w_j} \leq 1$ iff $\delta \leq (1 - y_j)w_j$
  - Recall $y_j < x_j$, so $y_j < 1$, which means $(1 - y_j) > 0$
  - So, this constrains $\delta$ to be smaller than some **positive number**

# FEASIBILITY OF $Y'$

- Finally, we show $weight(Y') \leq M$




- Recall changes to get $Y'$ from $Y$
  - We move $\delta$ weight from item $k$ to item $j$
  - This does not change the total weight!
- So $weight(Y') = weight(Y) \leq M$
- Therefore, $Y'$ **is feasible!**

# SUPERIORITY OF $Y'$

- Finally we compute $profit(Y')$
- $profit(Y') = profit(Y) + \frac{\delta}{w_j} p_j - \frac{\delta}{w_k} p_k$
- $= profit(Y) + \delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right)$
- Since j is before k, and we consider items with more profit per unit weight first, we have $\frac{p_j}{w_j} > \frac{p_k}{w_k}$.
- So, if $\delta > 0$ then $\delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right) > 0$
- Since we can choose $\delta > 0$, we have $profit(Y') > profit(Y)$.

(Fraction of item j **added**)
× (profit for item j)

(Fraction of item k **removed**)
× (profit for item k)

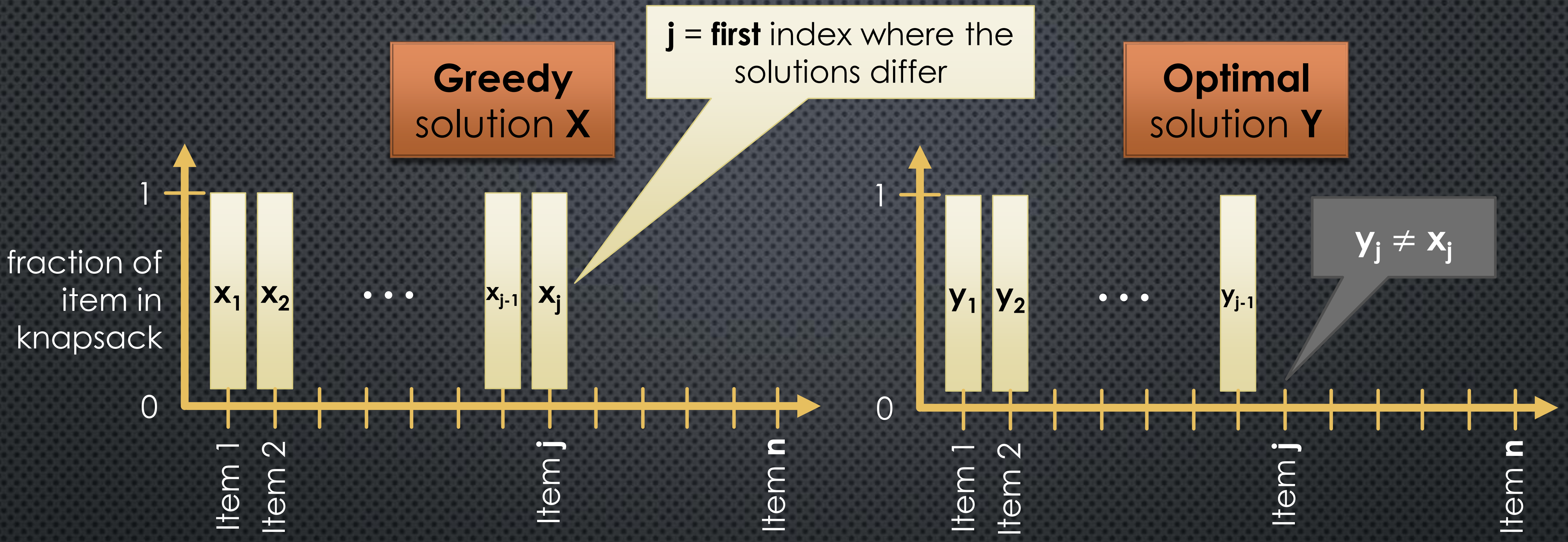
Contradicts optimality of Y!
So assumption $X \neq Y$ is bad.
Therefore, X is optimal.

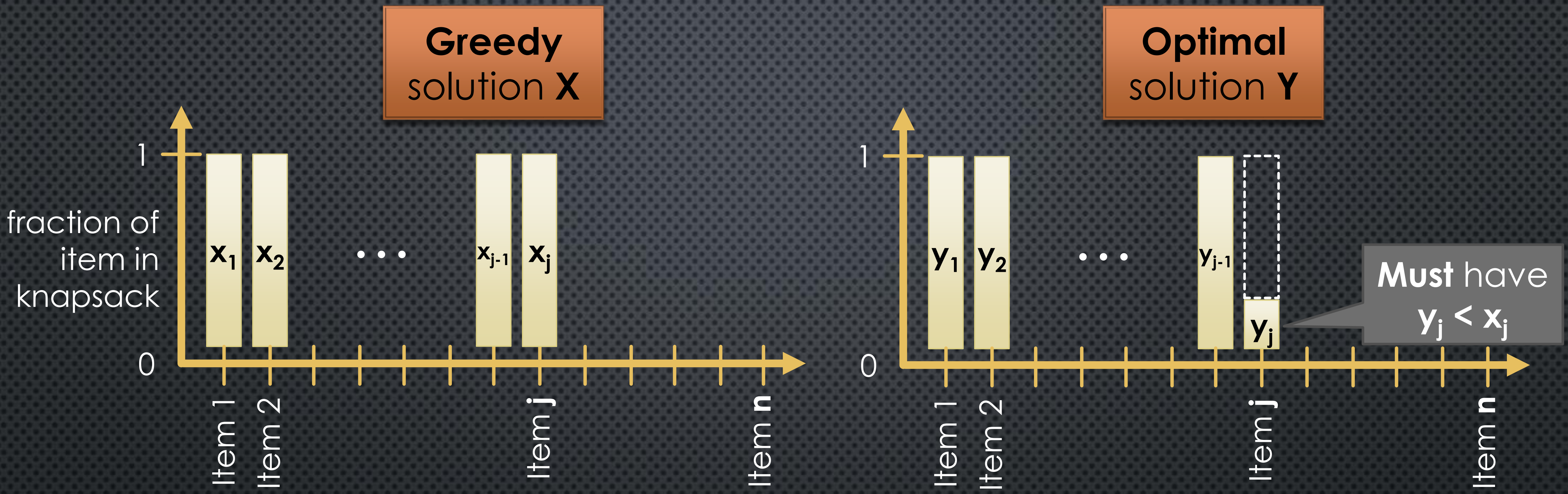
Covering the next 9 slides is homework!

# WHAT IF ELEMENTS DON'T HAVE DISTINCT PROFIT/WEIGHT RATIOS?

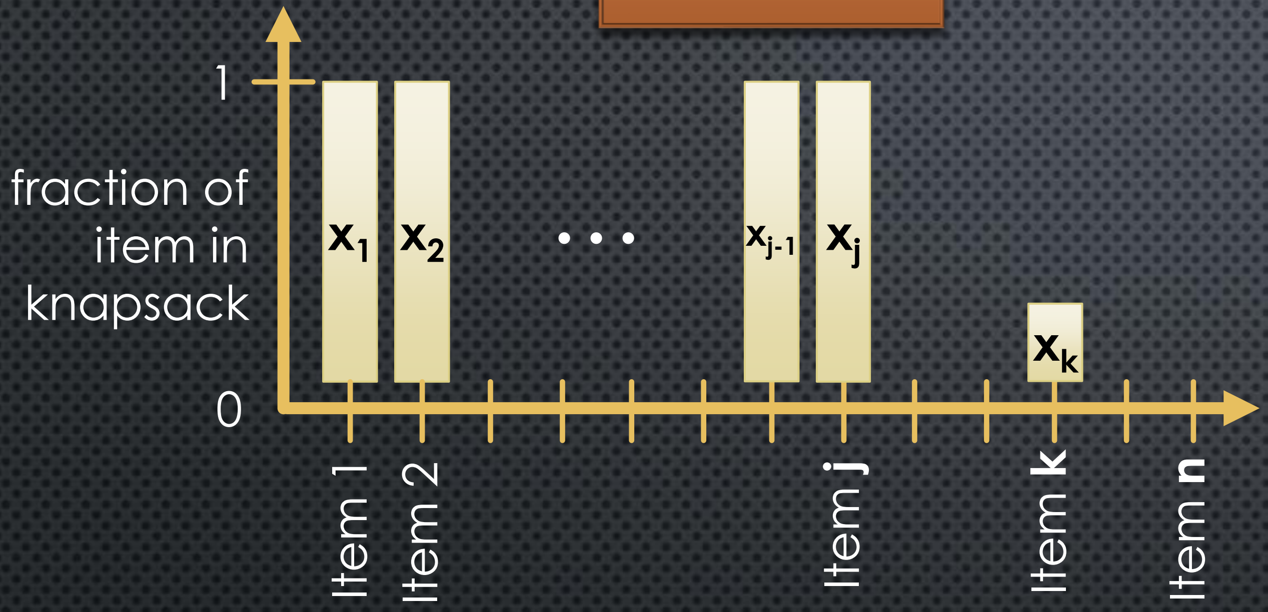
# OPTIMALITY PROOF WITHOUT DISTINCTNESS

- There may be many optimal solutions
- **Key idea:** Let $Y$ be an optimal solution
  that **matches $X$ on a maximal number of indices**
- **Observe**: if $X$ is really optimal, then $Y = X$
- Suppose not for contra
  - We will modify $Y$, preserving its optimality,
    but making it match $X$ on **one more index** (a contradiction!)


Greedy solution X
j = first index where the solutions differ
Optimal solution Y
1
0
fraction of item in knapsack
$x_1$ $x_2$ … $x_{j-1}$ $x_j$
Item 1
Item 2
Item j
Item n
$y_1$ $y_2$ … $y_{j-1}$
$y_j \neq x_j$


Greedy solution X
fraction of item in knapsack
1
0
$x_1$ $x_2$ . . . $x_{j-1}$ $x_j$
Item 1
Item 2
Item j
Item n
Optimal solution Y
1
0
$y_1$ $y_2$ . . . $y_{j-1}$ $y_j$
Item 1
Item 2
Item j
Item n
Must have $y_j < x_j$

**Greedy solution X**




Must exist **k > j** such that **$y_k$ > $x_k$** because weight of $X$ and $Y$ must be the same

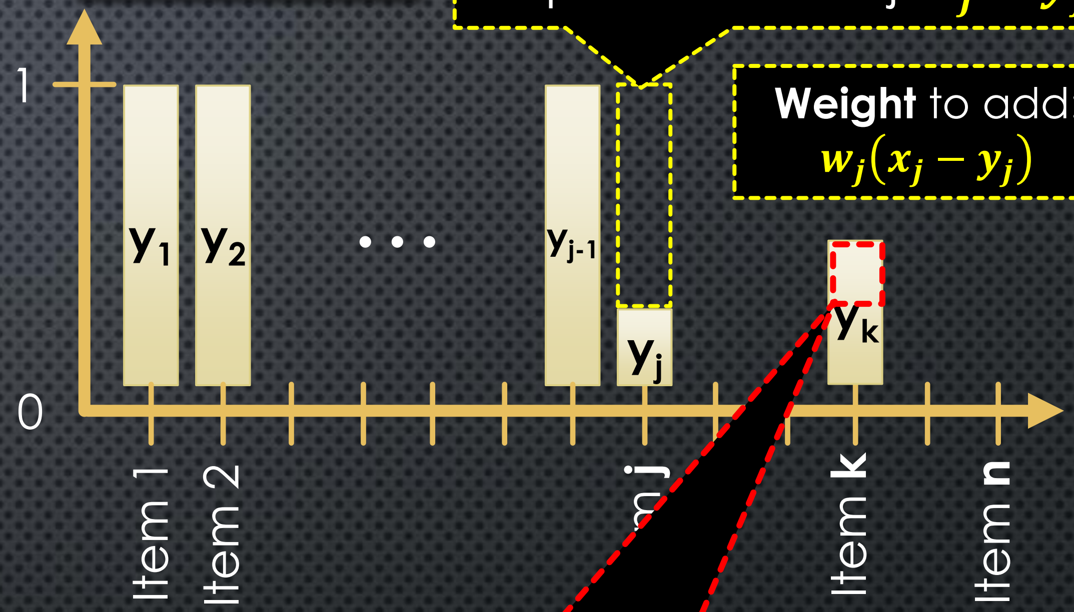
**Remove** some **weight** $\delta$ of item **k** and **add** the same weight of item **j**

With the goal of making the solutions **equal on index k <u>or</u> index j**

**Optimal solution Y**

**Fraction** we should **add** to j to make solutions equal on index j: $x_j - y_j$

**Weight** to add: $w_j(x_j - y_j)$



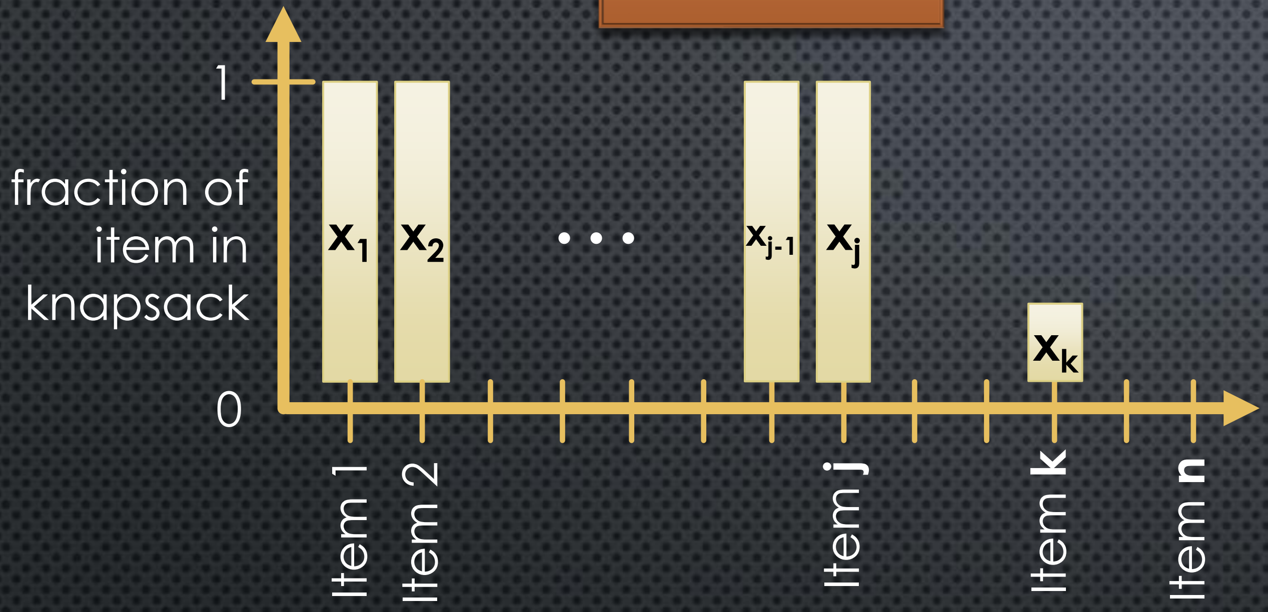

**Fraction** we should **remove** from k to make solutions equal on index k: $y_k - x_k$

**Weight** to remove: $w_k(y_k - x_k)$

Let $\delta = \min\{w_j(x_j - y_j), w_k(y_k - x_k)\}$

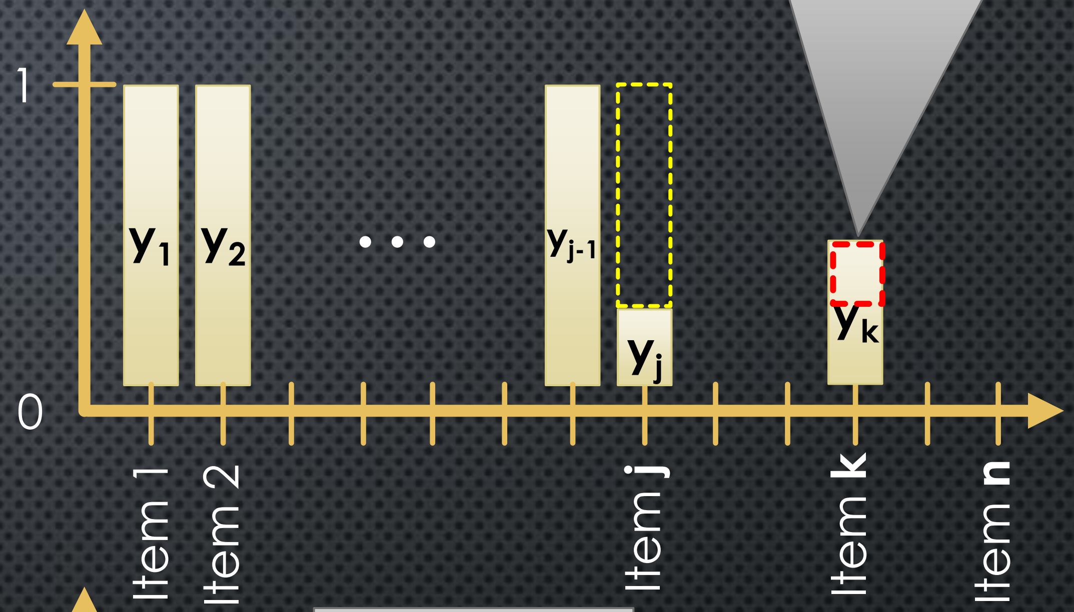
Observe $\delta > 0$

## Greedy solution X

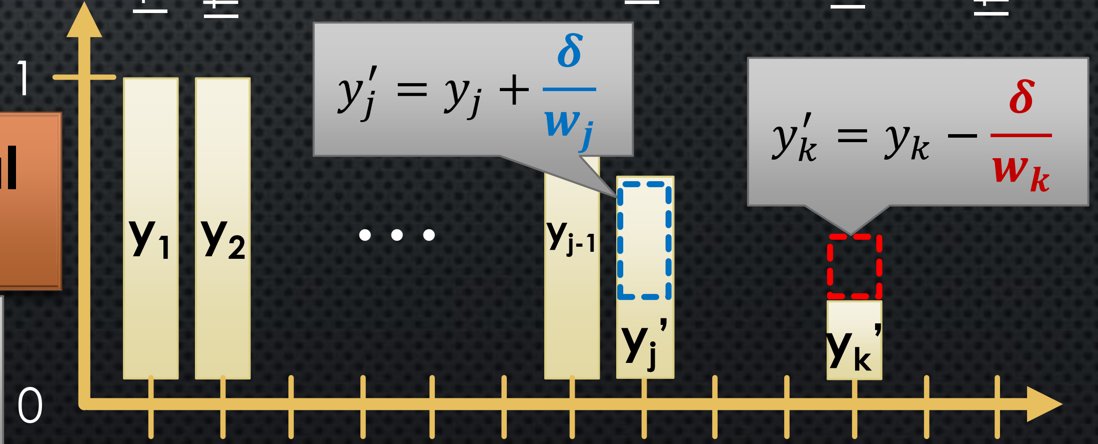



## Optimal solution Y

Suppose $\delta = w_k(y_k - x_k)$



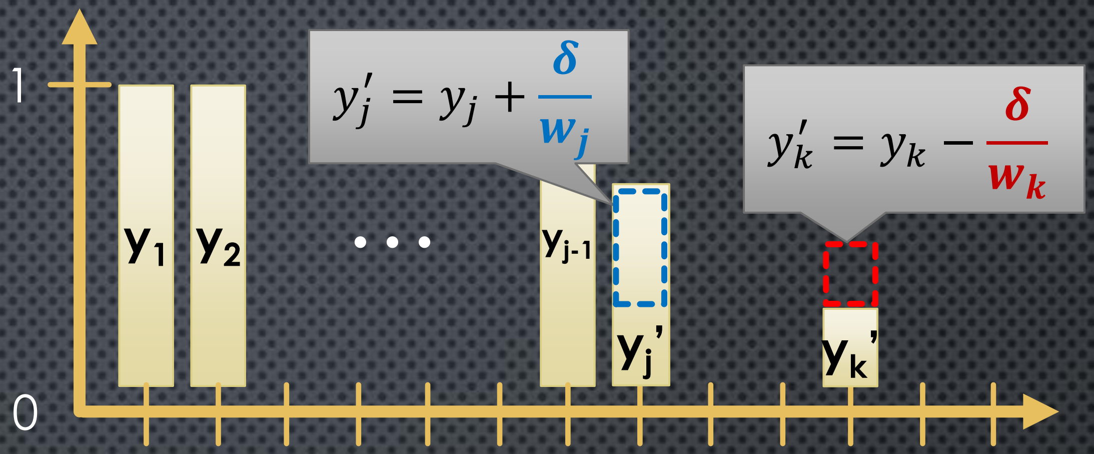

## Modified optimal solution Y'




In this case, since $\delta = w_k(y_k - x_k)$, we end up with $y'_k = x_k$

If $\delta$ were $w_j(x_j - y_j)$, we would have $y'_j = x_j$

**Modified optimal solution Y'**




To show $Y'$ is feasible, we show $weight(Y') \leq M$ and $y_k' \geq 0, y_j' \leq 1$

| **Weight** | We move $\delta$ weight from item $k$ to item $j$<br>This does not change the total weight!<br>So $weight(Y') = weight(Y) = M$ |
|---|---|

# FEASIBILITY OF $Y'$

- Showing $y'_k \geq 0$
  - By definition, $y'_k = y_k - \frac{\delta}{w_k} \geq 0$ iff $\boldsymbol{\delta \leq y_k w_k}$
  - But $\delta$ is the **minimum** of $w_j(x_j - y_j)$ and $w_k(y_k - x_k) \leq w_k y_k$
  - And $w_k(y_k - x_k) \leq w_k y_k$ **so** $\boldsymbol{\delta \leq y_k w_k}$
- Showing $y'_j \leq 1$
  - $y'_j = y_j + \frac{\delta}{w_j} \leq 1$ iff $\frac{\delta}{w_j} \leq 1 - y_j$ **iff** $\boldsymbol{\delta \leq w_j(1 - y_j)}$ (rearranging)
  - $\delta \leq \boldsymbol{w_j(x_j - y_j)}$ (definition of $\delta$)
  - and $w_j(x_j - y_j) \leq \boldsymbol{w_j(1 - y_j)}$ (by feasibility of X, i.e., $x_j \leq 1$)

# PROFIT OF $Y'$

(Fraction of item j **added**) × (profit for entire item)

- $profit(Y') = profit(Y) + \frac{\delta}{w_j} p_j - \frac{\delta}{w_k} p_k = profit(Y) + \delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right)$
- Since j is before k, and we consider items with more profit per unit weight first, we have $\frac{p_j}{w_j} \geq \frac{p_k}{w_k}$.
- Since $\delta > 0$ and $\frac{p_j}{w_j} \geq \frac{p_k}{w_k}$, we have $\delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right) \geq 0$
- Since $Y$ is optimal, this **cannot be positive**
- So $Y'$ is a new optimal solution
  that **matches $X$ on one more index than $Y$**
- Contradiction: $Y$ matched $X$ on a **maximal** number of indices!

# SUMMARIZING EXCHANGE ARGUMENTS

- If inputs are distinct
  - So there is a unique optimal solution
  - Let O != G be an optimal solution that beats greedy
  - Show how to change O to obtain a better solution
- If not
  - There may be many optimal solutions
  - Let O != G be an optimal solution that matches greedy on as many choices as possible
  - Show how to change O to obtain an optimal solution O’ that matches greedy for even more choices