CS 341: ALGORITHMS

Lecture 6: greedy algorithms II

Readings: see website

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OPTIMALITY PROOF

for greedy interval selection

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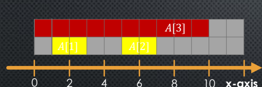
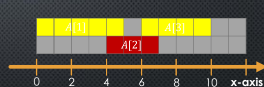
Goal: choose **as many** disjoint intervals as possible, (i.e., without any overlap)

Algorithm: Sort the intervals in increasing order of finishing times. At any stage, choose the earliest finishing interval that is disjoint from all previously chosen intervals (i.e., the local evaluation criterion is f_i).





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PROVING OPTIMALITY

- Consider an input A[1..n]
- Let **G** be the greedy solution
- Let **O** be an optimal solution
- "Greedy stays ahead" argument
  - Intuition: out of the a given set of intervals, greedy picks **as many as optimal**

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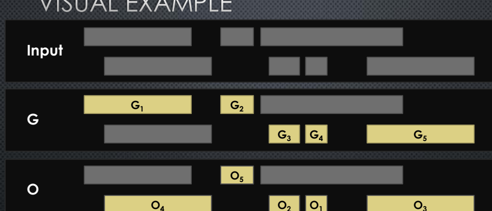
VISUAL EXAMPLE



How to compare G and O? **Imagine <u>reordering</u> O to match G!**

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<u>CRUCIAL:</u> We are **NOT** assuming the optimal **algorithm** uses the same sort order!

We are merely **imagining reordering** the intervals chosen by the optimal algorithm so we can easily **compare their finish times** to intervals in **G**

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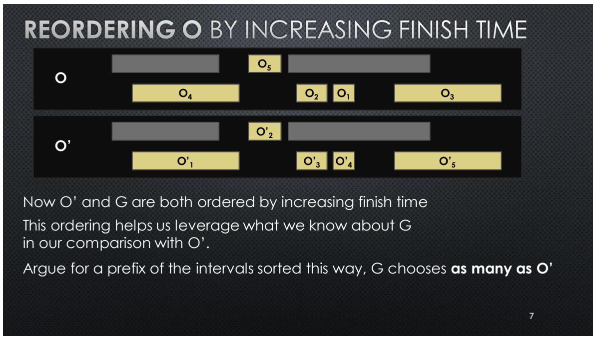
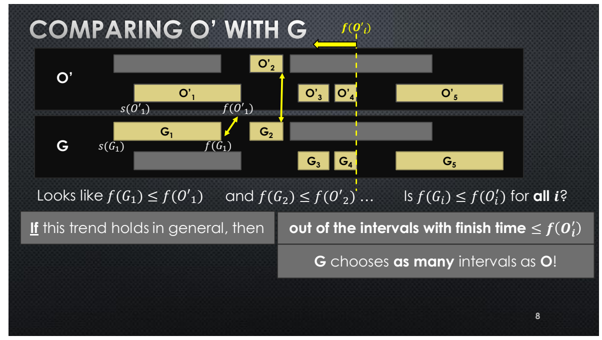

REORDERING O BY INCREASING FINISH TIME
O
O5
O4
O2
O1
O3
O'
O'2
O'1
O'3
O'4
O'5
Now O' and G are both ordered by increasing finish time
This ordering helps us leverage what we know about G in our comparison with O'.
Argue for a prefix of the intervals sorted this way, G chooses as many as O'
7


COMPARING O' WITH G
$f(O'_i)$
O'
O'2
O'1
O'3
O'4
O'5
$s(O'_1)$
$f(O'_1)$
G
G1
G2
$s(G_1)$
$f(G_1)$
G3
G4
G5
Looks like $f(G_1) \le f(O'_1)$ and $f(G_2) \le f(O'_2)$ … Is $f(G_i) \le f(O'_i)$ for all i?
If this trend holds in general, then
out of the intervals with finish time $\le f(O'_i)$
G chooses as many intervals as O!
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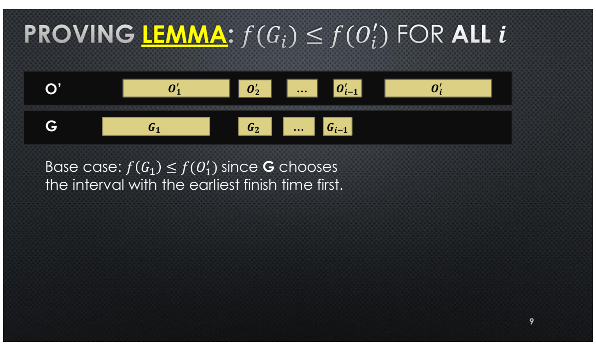
PROVING LEMMA: $f(G_i) \le f(O'_i)$ FOR ALL i
O'
O'_1
O'_2
…
O'_{i-1}
O'_i
G
G_1
G_2
…
G_{i-1}
Base case: $f(G_1) \le f(O'_1)$ since G chooses the interval with the earliest finish time first.
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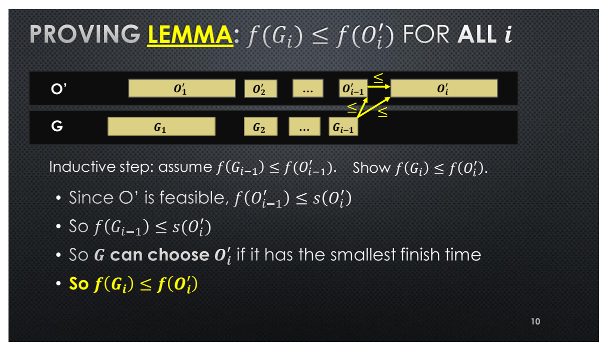
PROVING LEMMA: $f(G_i) \le f(O'_i)$ FOR ALL i
O'
O'_1
O'_2
…
O'_{i-1}
O'_i
G
G_1
G_2
…
G_{i-1}
Inductive step: assume $f(G_{i-1}) \le f(O'_{i-1})$. Show $f(G_i) \le f(O'_i)$.
Since O' is feasible, $f(O'_{i-1}) \le s(O'_i)$
So $f(G_{i-1}) \le s(O'_i)$
So G can choose O'_i if it has the smallest finish time
So $f(G_i) \le f(O'_i)$
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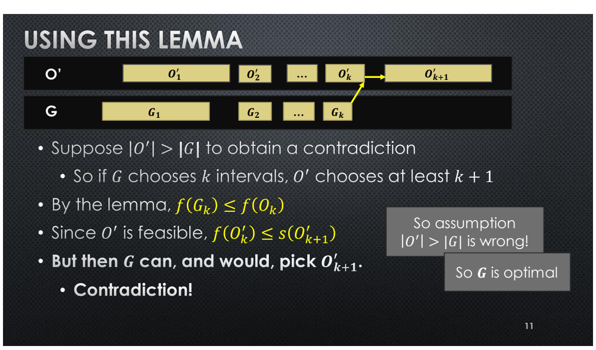
USING THIS LEMMA
O'
O'_1
O'_2
…
O'_k
O'_{k+1}
G
G_1
G_2
…
G_k
Suppose $|O'| > |G|$ to obtain a contradiction
So if G chooses k intervals, O' chooses at least $k+1$
By the lemma, $f(G_k) \le f(O_k)$
Since O' is feasible, $f(O'_k) \le s(O'_{k+1})$
But then G can, and would, pick O'_{k+1}.
Contradiction!
So assumption $|O'| > |G|$ is wrong!
So G is optimal
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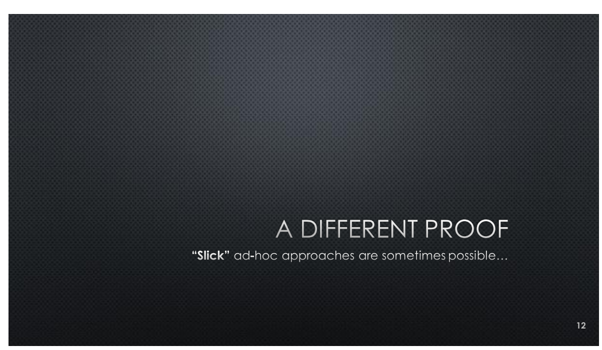
A DIFFERENT PROOF
"Slick" ad-hoc approaches are sometimes possible…
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Let $F = \{f_{i_1}, \ldots, f_{i_k}\}$ be the finishing times of the intervals in X

No interval finishes strictly to the left

f_{i_1} f_{i_2} f_{i_k}

No interval starts strictly to the right

greedy A_{i_1} A_{i_2} … A_{i_k}

No interval in is strictly between these points!

Would be chosen by greedy! (contradiction)

So, in addition to the intervals in X, only the following types of intervals are **possible**

Contains f_{i_1}

Contains f_{i_2}

Contains f_{i_1} **and** f_{i_2}

Thus, **every interval** contains some finishing time in F

And, two intervals in O **cannot contain the same** element of F

So, there must be as many finishing times in F as there are intervals in O. QED

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KNAPSACK PROBLEMS

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Problem 4.4

Knapsack

Instance: **Profits** $P = [p_1, \ldots, p_n]$; **weights** $W = [w_1, \ldots, w_n]$; *and a* **capacity**, M. *These are all positive integers.*

Feasible solution: *An n-tuple $X = [x_1, \ldots, x_n]$ where $\sum_{i=1}^{n} w_i x_i \leq M$.*

Gotta respect the **weight limit M…**

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Problem 4.4

Knapsack

Instance: **Profits** $P = [p_1, \ldots, p_n]$; **weights** $W = [w_1, \ldots, w_n]$; *and a* **capacity**, M. *These are all positive integers.*

Feasible solution: *An n-tuple $X = [x_1, \ldots, x_n]$ where $\sum_{i=1}^{n} w_i x_i \leq M$.*

In the **0-1 Knapsack** *problem (often denoted just as* **Knapsack***), we require that $x_i \in \{0, 1\}$, $1 \leq i \leq n$.*

In the **Rational Knapsack** *problem, we require that $x_i \in \mathbb{Q}$ and $0 \leq x_i \leq 1$, $1 \leq i \leq n$.*

Find: *A feasible solution X that maximizes $\sum_{i=1}^{n} p_i x_i$.*

0-1 Knapsack: NP Hard. Probably requires exponential time to solve…

Rational knapsack: Can be solved in polynomial time by a greedy alg!

Lets discuss this now… other one later

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POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 1:** consider items in **decreasing** order of **profit** (i.e., we maximize the local evaluation criterion p_i)
- Let's try an example input
  - Profits $P = [20, 50, \mathbf{100}]$
  - Weights $W = [10, 20, 10]$
  - Weight limit $M = 10$
- Algorithm selects last item for 100 profit
  - Looks optimal in this example

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POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 1:** consider items in **decreasing** order of **profit** (i.e., we maximize the local evaluation criterion p_i)
- How about a **second example input**
  - Profits $P = [20, 50, \mathbf{100}]$
  - Weights $W = [10, 20, 100]$
  - Weight limit $M = 10$
- Algorithm selects last item for **10** profit
  - **Not optimal!**

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POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 2:** consider items in **increasing** order of **weight** (i.e., we minimize the local evaluation criterion w_i)
- **Counterexample**
  - Profits $P = [20,50,100]$
  - Weights $W = [\mathbf{10}, 20,100]$
  - Weight limit $M = 10$
- Algorithm selects first item for 20 profit
  - It **could** select half of second item, for 25 profit!

POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 3:** consider items in **decreasing** order of **profit divided by weight** (i.e., we maximize local evaluation criterion p_i/w_i)
- Let's try our first example input
  - Profits $P = [20,50,\mathbf{100}]$
  - Weights $W = [10,20,10]$
  - Weight limit $M = 10$
- Profit divided by weight
  - $P/W = [2, 2.5, 10]$
- Algorithm selects last item for 100 profit (optimal)

POSSIBLE GREEDY STRATEGIES FOR KNAPSACK PROBLEMS

- **Strategy 3:** consider items in **decreasing** order of **profit divided by weight** (i.e., we maximize local evaluation criterion p_i/w_i)
- Let's try our second example input
  - Profits $P = [20, 50, \mathbf{100}]$
  - Weights $W = [10,20,100]$
  - Weight limit $M = 10$
- Profit divided by weight
  - $P/W = [2, 2.5, 1]$
- Algorithm selects second item for 25 profit (optimal)

It turns out strategy #3 **is** optimal...


```
Preprocess(A[1..n], M) // A[i] = (p_i, w_i)
    sort A by decreasing profit divided by weight
    let p[1..n] be the profits in A
    let w[1..n] be the weights in A
    return GreedyRationalKnapsack(p, w, M)

GreedyRationalKnapsack(p[1..n], w[1..n], M)
    X = [0, ..., 0]
    weight = 0

    for i = 1..n
        if weight + w[i] > M then
            X[i] = (M - weight) / w[i]
            break
        else
            X[i] = 1
            weight = weight + w[i]

    return X
```

No items are chosen yet

Current weight of knapsack

For all items

If we **cannot** fit the entire item

Put in as much of the item as you can, to **exactly fill** the knapsack

Otherwise take the entire item

Either X=(1,1,...,1,0,...,0) or X=(1,1,...,1,x_i,0,...,0) where $x_i \in (0,1)$


```
Preprocess(A[1..n], M) // A[i] = (p_i, w_i)
    sort A by decreasing profit divided by weight
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    return GreedyRationalKnapsack(p, w, M)

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    for i = 1..n
        if weight + w[i] > M then
            X[i] = (M - weight) / w[i]
            break
        else
            X[i] = 1
            weight = weight + w[i]

    return X
```

Running time complexity?

Can do preprocessing in $\Theta(n \log n)$

Create array in $\Theta(n)$ time

$\Theta(n)$ iterations each doing $\Theta(1)$ work

Total $\Theta(n \log n)$ (or $\Theta(n)$ if input is already sorted)

INFORMAL FEASIBILITY ARGUMENT

(SHOULD BE GOOD ENOUGH TO SHOW FEASIBILITY ON ASSESSMENTS)

- **Feasibility: all x_i are in $[0, 1]$ and total weight is $\leq M$**
- Either *everything* fits in the knapsack, or:
- When we exit the loop, **weight is exactly M**
- Every time we write to x_i it's either 0, 1 or $(M - weight)/w_i$ where $weight + w[i] > M$
  - Rearranging the latter we get $(M - weight)/w_i < 1$
  - And $weight \leq M$, so $(M - weight)/w_i \geq 0$
  - **So, we have $x_i \in [0, 1]$**

```
    for i = 1..n
        if weight + w[i] > M then
            X[i] = (M - weight) / w[i]
            break
        else
            X[i] = 1
            weight = weight + w[i]
```

MINOR MODIFICATION TO FACILITATE **FORMAL** PROOF

```
1   GreedyRationalKnapsack(p[1..n], w[1..n], M)
2       X = [0, ..., 0]
3       weight = 0
4
5       for i = 1..n
6           if weight + w[i] > M then
7               X[i] = (M - weight) / w[i]
8               weight = M
9               break
10          else
11              X[i] = 1
12              weight = weight + w[i]
13
14      return X
```

Does NOT change behaviour of the algorithm at all!

Optional slide, just for your notes

FORMAL FEASIBILITY ARG


```
5       for i = 1..n
6           if weight + w[i] > M then
7               X[i] = (M - weight) / w[i]
8               weight = M
9               break
10          else
11              X[i] = 1
12              weight = weight + w[i]
```


- **Loop invariant:** $\forall_i : x_i \in [0,1]$
- **and** $weight = \sum_{i=1}^{n} w_i x_i \leq M$
  - Base case. Initially weight = 0 and $\forall_i : x_i = 0$.
    - So $0 = weight = \sum_{i=1}^{n} w_i \cdot 0 = \sum_{i=1}^{n} w_i x_i \leq M$
  - Inductive step.
    - Suppose invariant holds at start of iteration i
    - Let $weight', x_i'$ denote values of $weight, x_i$ at **end** of iteration i
    - Prove invariant holds at end of iteration i
    - i.e., $\forall_i : x_i' \in [0,1]$ **and** $weight' = \sum_{i=1}^{n} w_i x'_i \leq M$

Optional slide, just for your notes

FORMAL FEASIBILITY ARG


```
5       for i = 1..n
6           if weight + w[i] > M then
7               X[i] = (M - weight) / w[i]
8               weight = M
9               break
10          else
11              X[i] = 1
12              weight = weight + w[i]
```


- **WTP:** $\forall_i : x_i' \in [0,1]$ **and** $weight' = \sum_{i=1}^{n} w_i x'_i \leq M$
- Case 1: $weight + w_i \leq M$
  - $x_i' = 1$ **which is in** $[0,1]$ (by line 11)
  - $weight' = weight + w_i$ (by line 12) and **this is** $\leq M$ by the case
  - $weight' = \sum_{k=1}^{n} x_k w_k + w_i$ (by invariant)
  - $weight' = \sum_{k=1}^{n} x_k w_k + x_i' w_i$ (since $x_i' = 1$)
  - And $x_k' = x_k$ for all $k \neq i$ and $x_i = 0$ so $\sum_{k=1}^{n} x_k' w_k = x_i' w_i + \sum_{k=1}^{n} x_k w_k$
  - Rearrange to get $\sum_{k=1}^{n} x_k w_k = (\sum_{k=1}^{n} x_k' w_k - x_i' w_i)$
  - So $weight' = (\sum_{k=1}^{n} x_k' w_k - x_i' w_i) + x_i' w_i = \sum_{k=1}^{n} x_k' w_k$

Optional slide, just for your notes

FORMAL FEASIBILITY ARG


```
5       for i = 1..n
6           if weight + w[i] > M then
7               X[i] = (M - weight) / w[i]
8               weight = M
9               break
10          else
11              X[i] = 1
12              weight = weight + w[i]
```


- **WTP:** $\forall_i : x_i' \in [0,1]$ **and** $weight' = \sum_{i=1}^{n} w_i x'_i \leq M$
- Case 2: $weight + w_i > M$
  - We have $w_i > M - weight$ (by case) and $M - weight \geq 0$ (by invariant)
  - So $0 \leq \frac{M - weight}{w_i} < 1$ **which means** $x_i' \in [0,1)$
  - $weight' = M = weight + (M - weight)$ (by line 8)
  - $weight' = \sum_{k=1}^{n} x_k w_k + (M - weight)$ (by invariant)
  - But $x_k' = x_k$ for all $k \neq i$ and $x_i = 0$ so $\sum_{k=1}^{n} x_k' w_k = x_i' w_i + \sum_{k=1}^{n} x_k w_k$
  - Rearrange to get $\sum_{k=1}^{n} x_k w_k = (\sum_{k=1}^{n} x_k' w_k - x_i' w_i)$
  - So $weight' = (\sum_{k=1}^{n} x_k' w_k - x_i' w_i) + (M - weight)$
  - And $M - weight = x_i' w_i$ **so** $weight' = \sum_{k=1}^{n} x_k' w_k$

Optional slide, just for your notes

EXCHANGE ARGUMENT
for proving optimality

OPTIMALITY – AN **EXCHANGE** ARUGMENT

For simplicity, assume that the profit / weight ratios are all distinct, so

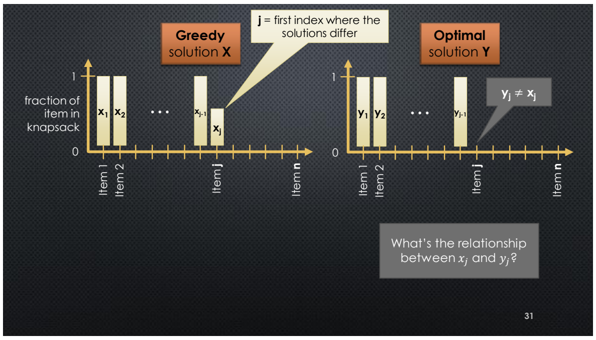
$$\frac{p_1}{w_1} > \frac{p_2}{w_2} > \cdots > \frac{p_n}{w_n}.$$

Suppose the greedy solution is $X = (x_1, \ldots, x_n)$ and the optimal solution is $Y = (y_1, \ldots, y_n)$.

We will prove that $X = Y$, i.e., $x_j = y_j$ for $j = 1, \ldots, n$. Therefore there is a unique optimal solution and it is equal to the greedy solution.

Suppose $X \neq Y$. — To obtain a contradiction

Pick the smallest integer j such that $x_j \neq y_j$. — X and Y are **identical** up to x_j and y_j, respectively



Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
x1 x2 … xj-1 xj
item 1 item 2 item j item n
y1 y2 … yj-1
yj ≠ xj
item 1 item 2 item j item n
What's the relationship between xj and yj?
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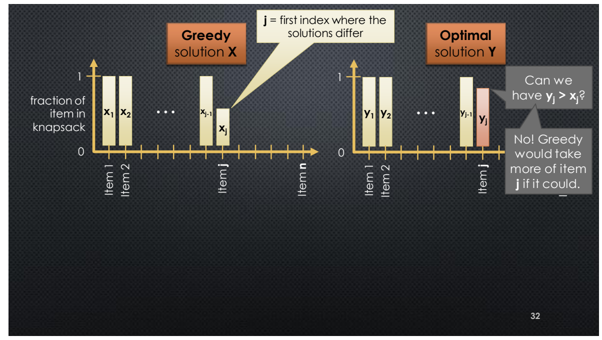

Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
x1 x2 … xj-1 xj
item 1 item 2 item j item n
y1 y2 … yj-1 yj
Can we have yj > xj?
No! Greedy would take more of item j if it could.
item 1 item 2 item j item n
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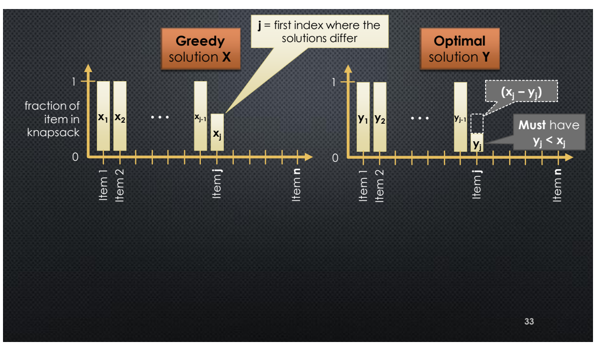

Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
x1 x2 … xj-1 xj
item 1 item 2 item j item n
y1 y2 … yj-1 yj
(xj – yj)
Must have yj < xj
item 1 item 2 item j item n
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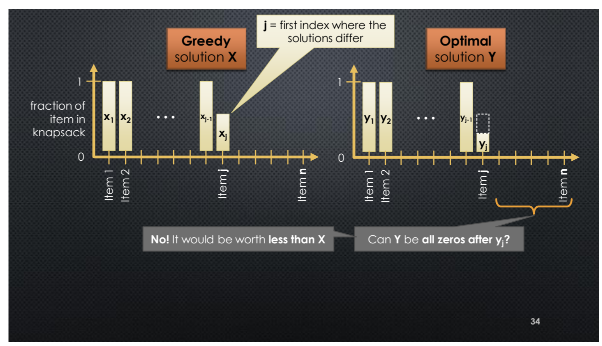

Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
x1 x2 … xj-1 xj
item 1 item 2 item j item n
y1 y2 … yj-1 yj
item 1 item 2 item j item n
No! It would be worth less than X
Can Y be all zeros after yj?
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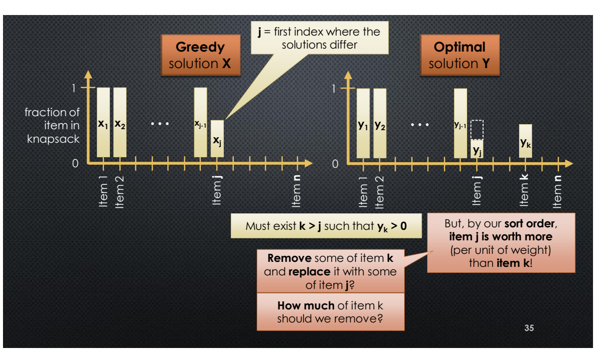

Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
x1 x2 … xj-1 xj
item 1 item 2 item j item n
y1 y2 … yj-1 yj yk
item 1 item 2 item j item k item n
Must exist k > j such that yk > 0
But, by our sort order, item j is worth more (per unit of weight) than item k!
Remove some of item k and replace it with some of item j?
How much of item k should we remove?
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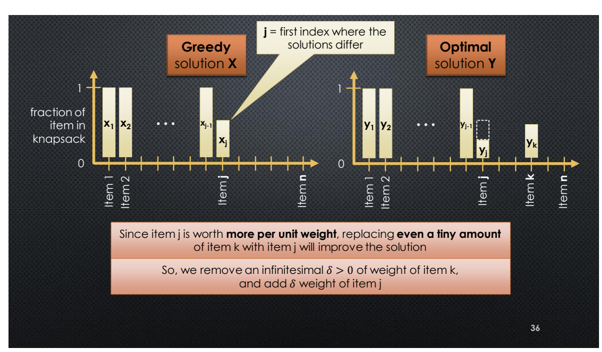

Greedy solution X
j = first index where the solutions differ
Optimal solution Y
fraction of item in knapsack
1
0
x1 x2 … xj-1 xj
item 1 item 2 item j item n
y1 y2 … yj-1 yj yk
item 1 item 2 item j item k item n
Since item j is worth more per unit weight, replacing even a tiny amount of item k with item j will improve the solution
So, we remove an infinitesimal δ > 0 of weight of item k, and add δ weight of item j
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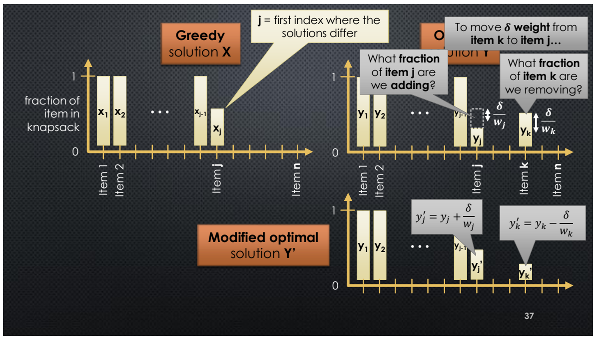

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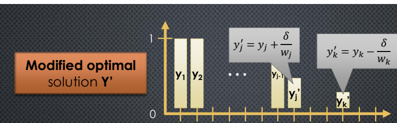

The idea is to show that

Y' is feasible, and

profit(Y') > profit(Y).

This contradicts the optimality of Y and proves that $X = Y$.

To show Y' is feasible, we show $y_k' \geq 0, y_j' \leq 1$ and $weight(Y') \leq M$

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FEASIBILITY OF Y'

- To show Y' is feasible, we show $y_k' \geq 0, y_j' \leq 1$ and $weight(Y') \leq M$
- Let's show $y_k' \geq 0$
  - By definition, $y_k' = y_k - \frac{\delta}{w_k}$
  - So, $y_k' \geq 0$ iff $y_k - \frac{\delta}{w_k} \geq 0$ iff $\delta \leq y_k w_k$
  - And we know y_k and w_k are both **positive**
  - So, this constrains δ to be smaller than this **positive number**
  - Therefore, it is possible to choose positive δ s.t. $y_k' \geq 0$

Existence proof, but a **non-constructive** one

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FEASIBILITY OF Y'

- To show Y' is feasible, we show $y_k' \geq 0, y_j' \leq 1$ and $weight(Y') \leq M$
- Now let's show $y_j' \leq 1$
  - By definition, $y_j' = y_j + \frac{\delta}{w_j}$
  - So, $y_j' \leq 1$ iff $y_j + \frac{\delta}{w_j} \leq 1$ iff $\delta \leq (1 - y_j)w_j$
  - Recall $y_j < x_j$, so $y_j < 1$, which means $(1 - y_j) > 0$
  - So, this constrains δ to be smaller than some **positive number**

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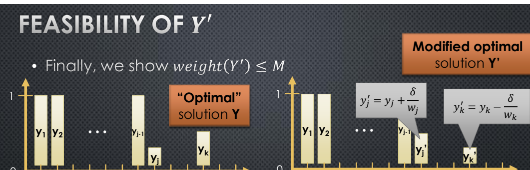
FEASIBILITY OF Y'

- Finally, we show $weight(Y') \leq M$



- Recall changes to get Y' from Y
  - We move δ weight from item k to item j
  - This does not change the total weight!
- So $weight(Y') = weight(Y) \leq M$
- Therefore, Y' **is feasible!**

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SUPERIORITY OF Y'

- Finally we compute $profit(Y')$
- $profit(Y') = profit(Y) + \frac{\delta}{w_j} p_j - \frac{\delta}{w_k} p_k$
- $= profit(Y) + \delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right)$
- Since j is before k, and we consider items with more profit per unit weight first, we have $\frac{p_j}{w_j} > \frac{p_k}{w_k}$.
- So, if $\delta > 0$ then $\delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right) > 0$
- Since we can choose $\delta > 0$, we have $profit(Y') > profit(Y)$.

(Fraction of item j **added**) × (profit for item j)

(Fraction of item k **removed**) × (profit for item k)

Contradicts optimality of Y! So assumption $X \neq Y$ is bad. Therefore, X is optimal.

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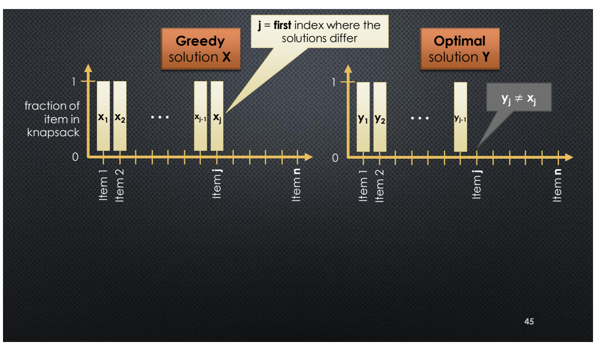
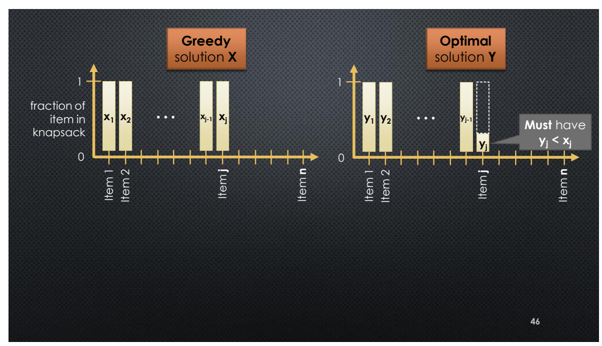
Covering the next 9 slides is homework!

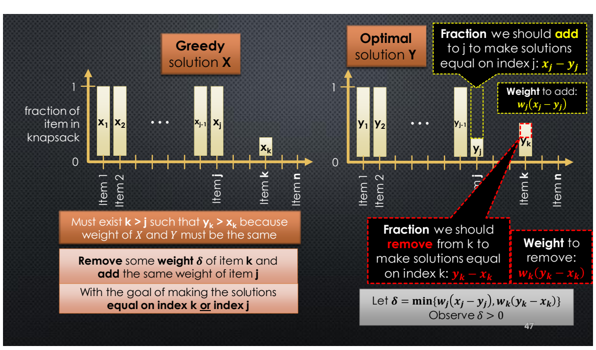
WHAT IF ELEMENTS DON'T HAVE DISTINCT PROFIT/WEIGHT RATIOS?

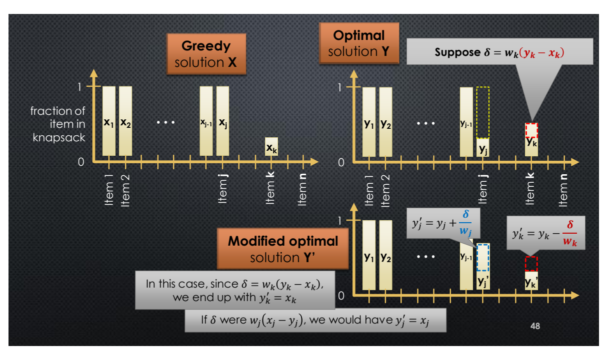
OPTIMALITY PROOF WITHOUT DISTINCTNESS

- There may be many optimal solutions
- **Key idea:** Let Y be an optimal solution that **matches X on a maximal number of indices**
- **Observe**: if X is really optimal, then $Y = X$
- Suppose not for contra
  - We will modify Y, preserving its optimality, but making it match X on **one more index** (a contradiction!)

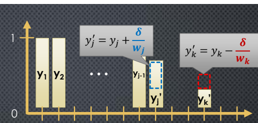








Modified optimal solution Y'



To show Y' is feasible, we show $weight(Y') \le M$ and $y'_k \ge 0, y'_j \le 1$

Weight
We move δ weight from item k to item j
This does not change the total weight!
So $weight(Y') = weight(Y) = M$

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FEASIBILITY OF Y'

- Showing $y'_k \ge 0$
  - By definition, $y'_k = y_k - \frac{\delta}{w_k} \ge 0$ iff $\boldsymbol{\delta \le y_k w_k}$
  - But δ is the **minimum** of $w_j(x_j - y_j)$ and $w_k(y_k - x_k) \le w_k y_k$
  - And $w_k(y_k - x_k) \le w_k y_k$ **so** $\boldsymbol{\delta \le y_k w_k}$
- Showing $y'_j \le 1$
  - $y'_j = y_j + \frac{\delta}{w_j} \le 1$ iff $\frac{\delta}{w_j} \le 1 - y_j$ **iff** $\boldsymbol{\delta \le w_j(1 - y_j)}$ (rearranging)
  - $\delta \le \boldsymbol{w_j(x_j - y_j)}$ (definition of δ)
  - and $w_j(x_j - y_j) \le \boldsymbol{w_j(1 - y_j)}$ (by feasibility of X, i.e., $x_j \le 1$)

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PROFIT OF Y'

(Fraction of item j **added**) × (profit for entire item)

- $profit(Y') = profit(Y) + \frac{\delta}{w_j} p_j - \frac{\delta}{w_k} p_k = profit(Y) + \delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right)$
- Since j is before k, and we consider items with more profit per unit weight first, we have $\frac{p_j}{w_j} \ge \frac{p_k}{w_k}$.
- Since $\delta > 0$ and $\frac{p_j}{w_j} \ge \frac{p_k}{w_k}$, we have $\delta\left(\frac{p_j}{w_j} - \frac{p_k}{w_k}\right) \ge 0$
- Since Y is optimal, this **cannot be positive**
- So Y' is a new optimal solution that **matches X on one more index than Y**
- Contradiction: Y matched X on a **maximal** number of indices!

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SUMMARIZING EXCHANGE ARGUMENTS

- If inputs are distinct
  - So there is a unique optimal solution
  - Let O != G be an optimal solution that beats greedy
  - Show how to change O to obtain a better solution
- If not
  - There may be many optimal solutions
  - Let O != G be an optimal solution that matches greedy on as many choices as possible
  - Show how to change O to obtain an optimal solution O' that matches greedy for even more choices

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