Lecture 4: Divide and Conquer III

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

September 19, 2023

Overview

- Closest Pair
- Non-dominated points
- Acknowledgements

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Unit cost model!
- Simplifying assumption: all x coordinates are distinct.
- **Exercise:** remove this assumption, but preserve the running time.

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$
- Can we do better?

Closest Pair

- **Input:** n points $(x_1, y_1), \dots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$

- Divide and conquer!
  1. Vertical line Λ that separates points into 2 halves (left and right of Λ)

     Use median finding algorithm from previous lecture.

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$

- Divide and conquer!
  1. Vertical line Λ that separates points into 2 halves (left and right of Λ)
     Use median finding algorithm from previous lecture.
  2. Let L and R be the set of points to left and right of Λ, respectively

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$

- Divide and conquer!
  1. Vertical line Λ that separates points into 2 halves (left and right of Λ)

     Use median finding algorithm from previous lecture.
  2. Let L and R be the set of points to left and right of Λ, respectively
  3. Solve closest pair for L and for R. Suppose the smallest distance δ is between points of L.

Are we done?

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$

- Divide and conquer!
  1. Vertical line Λ that separates points into 2 halves (left and right of Λ)

     Use median finding algorithm from previous lecture.
  2. Let L and R be the set of points to left and right of Λ, respectively
  3. Solve closest pair for L and for R. Suppose the smallest distance δ is between points of L.

     Are we done?

     Nope. Need to check if smallest distance is between points crossing from L to R.

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$

- Divide and conquer!
  1. Vertical line Λ that separates points into 2 halves (left and right of Λ)

     Use median finding algorithm from previous lecture.
  2. Let L and R be the set of points to left and right of Λ, respectively
  3. Solve closest pair for L and for R. Suppose the smallest distance δ is between points of L.

     Are we done?

     Nope. Need to check if smallest distance is between points crossing from L to R.

     Checking crossing pairs seems as hard as the original problem!

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \leq i < j \leq n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$

- Divide and conquer!
  1. Vertical line Λ that separates points into 2 halves (left and right of Λ)

     Use median finding algorithm from previous lecture.
  2. Let L and R be the set of points to left and right of Λ, respectively
  3. Solve closest pair for L and for R. Suppose the smallest distance δ is between points of L.

     Observation: only need to check if $\exists$ crossing pair with distance $< \delta$

Closest Pair

- **Input:** n points $(x_1, y_1), \ldots, (x_n, y_n) \in \mathbb{R}^2$
- **Output:** indices $1 \le i < j \le n$ which minimizes the distance
- Exhaustive search: compute all distances and output minimum one - running time $O(n^2)$

- Divide and conquer!
  1. Vertical line Λ that separates points into 2 halves (left and right of Λ)

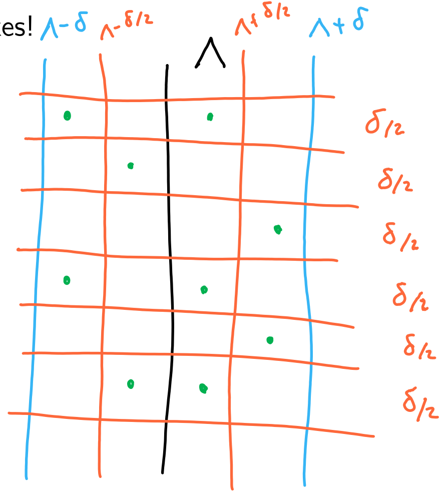
     Use median finding algorithm from previous lecture.
  2. Let L and R be the set of points to left and right of Λ, respectively
  3. Solve closest pair for L and for R. Suppose the smallest distance δ is between points of L.

     Observation: only need to check if $\exists$ crossing pair with distance $< \delta$

     Could just pay attention to points with x-coordinate within δ to line Λ... but still all points can be there...

Closest pair - boxing up

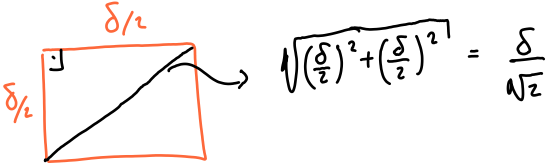
- Make $\delta/2 \times \delta/2$ boxes!



Closest pair - boxing up

- Make $\delta/2 \times \delta/2$ boxes!
- Each square box has ≤ 1 point from our set

Maximum distance inside square is $\delta/\sqrt{2}$



Closest pair - boxing up

- Make $\delta/2 \times \delta/2$ boxes!
- Each square box has ≤ 1 point from our set

  Maximum distance inside square is $\delta/\sqrt{2}$
- Each point only needs to compute distances with points within two horizontal layers

  All other distances are $> \delta$

Closest pair - boxing up

- Make $\delta/2 \times \delta/2$ boxes!
- Each square box has ≤ 1 point from our set

  Maximum distance inside square is $\delta/\sqrt{2}$
- Each point only needs to compute distances with points within two horizontal layers

  All other distances are $> \delta$
- Hence, each point needs only check its distance with ≤ 11 other points!

  Now we only need to check $O(n)$ pairs[1]

[1] Before boxing needed to check $\Omega(n^2)$ pairs

Algorithm

1. Find vertical line Λ
2. Recursively solve L, R subproblems
3. Linear scan to remove points $> \delta$ far (horizontally) from Λ
4. Sort points by y-coordinate, store them in array A
5. For each point in A, compute distances to next 11 points in A
6. Return minimum distance found.

Algorithm

1. Find vertical line Λ
2. Recursively solve L, R subproblems
3. Linear scan to remove points $> \delta$ far (horizontally) from Λ
4. Sort points by y-coordinate, store them in array A
5. For each point in A, compute distances to next 11 points in A
6. Return minimum distance found.

- **Correctness:** by arguments in previous slides.

Algorithm

1. Find vertical line Λ
2. Recursively solve L, R subproblems
3. Linear scan to remove points $> \delta$ far (horizontally) from Λ
4. Sort points by y-coordinate, store them in array A
5. For each point in A, compute distances to next 11 points in A
6. Return minimum distance found.

- **Correctness:** by arguments in previous slides.
- **Running time:** (naive)

$$T(n) = 2T(n/2) + O(n \log n) \Rightarrow T(n) = O(n \log^2 n)$$

Algorithm

1. Find vertical line Λ
2. Recursively solve L, R subproblems
3. Linear scan to remove points $> \delta$ far (horizontally) from Λ
4. Sort points by y-coordinate, store them in array A
5. For each point in A, compute distances to next 11 points in A
6. Return minimum distance found.

- **Correctness:** by arguments in previous slides.
- **Running time:** (sorting in beginning)
  We can first sort y-coordinates prior to recursing, and this sorted array can still be used in recursion. Thus, running time (with sorted input):

$$T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

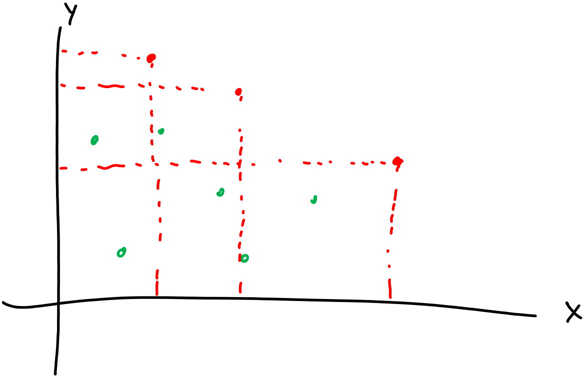
adding the time to sort doesn't change total runtime.

- Closest Pair
- Non-dominated points
- Acknowledgements

Non-dominated points

- Given two points (x_1, y_1) and (x_2, y_2)

$$(x_1, y_1) \text{ \textit{dominates} } (x_2, y_2) \text{ if } x_1 > x_2 \textbf{ and } y_1 > y_2.$$



Non-dominated points

- **Input:** set of n points $S := \{(x_1, y_1), \dots, (x_n, y_n)\}$
- **Output:** all *non-dominated* points of S
- **Model:** unit-cost model
- **Assumptions:** (for simplicity) distinct x values

Non-dominated points

- **Input:** set of n points $S := \{(x_1, y_1), \ldots, (x_n, y_n)\}$
- **Output:** all *non-dominated* points of S
- Naive algorithm:

For each point (x_i, y_i) check against all other points, if it is dominated or not.

Running time: $O(n^2)$

Non-dominated points

- **Input:** set of n points $S := \{(x_1, y_1), \ldots, (x_n, y_n)\}$
- **Output:** all *non-dominated* points of S
- Naive algorithm:

  For each point (x_i, y_i) check against all other points, if it is dominated or not.

  Running time: $O(n^2)$

- Can we do better?

Non-dominated points

- **Input:** set of n points $S := \{(x_1, y_1), \ldots, (x_n, y_n)\}$
- **Output:** all *non-dominated* points of S
- Naive algorithm:

  For each point (x_i, y_i) check against all other points, if it is dominated or not.

  Running time: $O(n^2)$
- Can we do better?
- Divide and conquer!
  1. Sort points according to x-coordinate
  2. Recursively solve two subproblems $n/2$ points to the left of middle (denoted S_L), $n/2$ points to the right of middle (denoted S_R)
  3. How do we combine?
     - (astute) Observation: no point in S_L dominates a point in S_R
     - Need to eliminate points from S_L which are dominated by a point in S_R
     - These must be the points with y-coordinate larger than the largest height of S_R!

Combining solutions to subproblems

- Let $ND_L = [P_1, \ldots, P_a]$ and $ND_R = [Q_1, \ldots, Q_b]$ be non-dominated points of S_L, S_R, respectively, *sorted by* x-coordinate.

Combining solutions to subproblems

- Let $ND_L = [P_1, \ldots, P_a]$ and $ND_R = [Q_1, \ldots, Q_b]$ be non-dominated points of S_L, S_R, respectively, *sorted by* x-coordinate.
- Must be the case that $y(Q_1) > y(Q_j)$ for all $j > 1$!

Combining solutions to subproblems

- Let $ND_L = [P_1, \ldots, P_a]$ and $ND_R = [Q_1, \ldots, Q_b]$ be non-dominated points of S_L, S_R, respectively, *sorted by* x-coordinate.
- Must be the case that $y(Q_1) > y(Q_j)$ for all $j > 1$!
- Thus, only need to compare $y(P_i)$ with $y_(Q_1)$!
- $O(n)$ time to combine!

Combining solutions to subproblems

- Let $ND_L = [P_1, \ldots, P_a]$ and $ND_R = [Q_1, \ldots, Q_b]$ be non-dominated points of S_L, S_R, respectively, *sorted by* x-coordinate.
- Must be the case that $y(Q_1) > y(Q_j)$ for all $j > 1$!
- Thus, only need to compare $y(P_i)$ with $y_(Q_1)$!
- $O(n)$ time to combine!

- Algorithm
  1. Sort points by x-coordinate
  2. Recursively solve two subproblems $n/2$ points to the left of middle (denoted S_L), $n/2$ points to the right of middle (denoted S_R)
  3. Combine points as above (linear scan)
  4. Output non-dominated points

Combining solutions to subproblems

- Let $ND_L = [P_1, \ldots, P_a]$ and $ND_R = [Q_1, \ldots, Q_b]$ be non-dominated points of S_L, S_R, respectively, *sorted by* x-coordinate.
- Must be the case that $y(Q_1) > y(Q_j)$ for all $j > 1$!
- Thus, only need to compare $y(P_i)$ with $y_(Q_1)$!
- $O(n)$ time to combine!

- Algorithm
  1. Sort points by x-coordinate
  2. Recursively solve two subproblems $n/2$ points to the left of middle (denoted S_L), $n/2$ points to the right of middle (denoted S_R)
  3. Combine points as above (linear scan)
  4. Output non-dominated points
- **Running time:**
  1. sorting $O(n \log n)$
  2. Recursion (for sorted input):

$$T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \log n)$$

  3. **Total runtime:** $O(n \log n)$

Acknowledgement

- Based on Prof. Lau's lecture 4

  `https://cs.uwaterloo.ca/~lapchi/cs341/notes/L04.pdf`
- Based on Prof. Brown's lecture (see course webpage)

References I

Kleinberg, John and Tardos, Eva (2006)
Algorithm Design.
Addison Wesley