Lecture 22: Intractability II
NP-Hardness

Rafael Oliveira

University of Waterloo
Cheriton School of Computer Science

rafael.oliveira.teaching@gmail.com

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Overview

- Navigating the world of P and NP
  - 2SAT
- Beyond decision problems: NP-hardness
  - NP-hard reductions
- Acknowledgements

Subtleties

Similar looking problems, wildly different complexity:

- **Hamilton Cycle:**
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- In general, we need to be careful when distinguishing or making reductions between problems.

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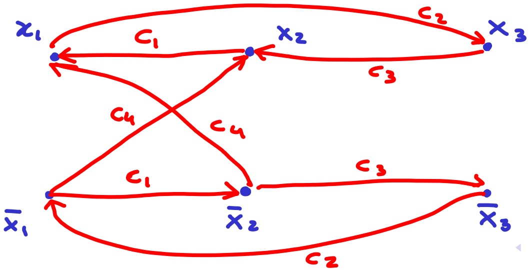

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- If for some $i \in [n]$, both x_i and $\overline{x_i}$ are bad, then return NO. Otherwise, return YES.

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- The above is our definition of *NP-hardness*:

Problem B is *NP-hard* if there is NP-complete problem A such that

$$A \leq_T B.$$

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- TSP-OPT:
  - **Input:** complete graph $G(V, E, d)$ where $d : E \to \mathbb{R}_{\geq 0}$
  - **Output:** hamiltonian cycle in G of minimum total distance

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Non-Trivial NP-hardness reduction

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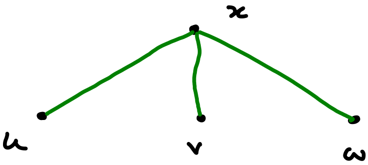

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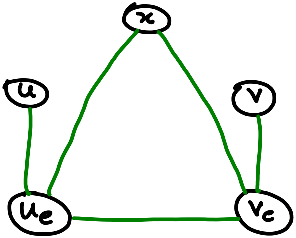

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  - Edge gadget H_e:
  - Let $H(U, F)$ be graph given by:
    - $U = V \sqcup \{x\} \sqcup \{u_e, v_e\}_{\{u,v\}=:e\in E}$
    - $F = \{\{x, w\}\}_{w\in U\setminus\{x\}} \sqcup \{\{u_e, v_e\}\}_{e\in E} \sqcup \{\{u, u_e\}, \{v, v_e\}\}_{\{u,v\}=:e\in E}$

Note that H *does not* have any edges from G

Proof of Correctness - Part 1

- **Claim 1:** G contains independent set $I \subset V$ with $|I| = k \Rightarrow$ there is cut $S \subset U$ in H such that

$$|\delta(S)| \geq k + 4 \cdot |E|$$

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2. For each edge $e = \{u, v\} \in E$ do
   - if $u \in I$, $v \notin I$, then add v_e to S
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   In all above cases, add four of five *edge gadget H_e* edges

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- For every $w \in I$, the edge $\{x, w\}$ is cut by S
- For every edge $\{u, v\} =: e \in E$, exactly 4 edges of H_e are cut.

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- **Claim 2:** Given cut $S \subset U$ in H with

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then G contains independent set $I \subset V$ of size $\geq k$.

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  - Letting $e(I)$ be number of edges between elements of I in G:

$$|\delta(S)| = |I| + \sum_{e \in E} |\delta_{H_e}(S)| \leq |I| + 3e(I) + 4(|E| - e(I)) = |I| + 4|E| - e(I)$$

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  - So for each $u, v \in I$ with $\{u, v\} \in E$, we can afford to remove one of the endpoints from S, decreasing $|I|$ by one. After $e(I)$ removals, get our independent set.

Acknowledgement

Based on

- [Erickson 2019, Chapter 12]
- Debmalya's Lecture 22

  https://courses.cs.duke.edu/fall19/compsci638/fall19_notes/lecture22.pdf

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