Module 08: Searching and Sorting Algorithms

Topics:
• Searching algorithms
• Sorting algorithms
Application: Searching a list

Suppose you have a list \( L \). How could you determine if a particular value is in that list, if \( L \) is in no particular order?

Algorithm (called Linear Search)

• Check the first element in \( L \): is it the one?
  – If Yes, return True
  – Else, check the next value

• The value is not in the list if you don’t find it (return False)
def linear_search (L, target):
    '''produces True if target is in L, False otherwise
    linear_search: (listof X) X -> Bool
    Note: equivalent to: target in L
    '''
    for val in L:
        if val == target:
            return True
    return False
Running Time of `linear_search`

• Let `n = len(L)`

• Best Case:
  – If target is in first position, we find it right away → $O(1)$

• Worst Case:
  – If target is not in $L$, we have to check all $n$ elements → $O(n)$
  – What is the other worst case?
Alternatives to Linear Search

• If $L$ is unsorted, we can’t do any better than Linear Search.

• How could we improve Linear Search if $L$ was sorted into increasing order?
  – Are there situations in which we could stop earlier?
  – Is this any faster in the worst case?
A better approach: Binary Search

- Suppose \( L \) is a listing of the taxpayers in Canada, sorted into increasing order by Social Insurance numbers.
- Approximately 22,000,000 entries
- Look at \( L[11000000] \)
- Is it the target taxpayer?
  - If yes, stop.
  - If not, is \( \text{target} < L[11000000] \) ?
    - If yes, then \( \text{target} \) is in the first half of \( L \)
    - If not, then \( \text{target} \) is in the second half of \( L \)
  - Repeat this process for the half containing \( \text{target} \)
Developing `binary_search`

- We need to determine how to keep track of the section of the list still being searched
  - Variables `beginning, end`
  - Determine their initial values
- Determine the `middle` position
- If `L[middle]` is target, return `stop`
- Otherwise, update `beginning` and `end`
- Determine when we to continue (or stop) searching
def binary_search(L, target):
    beginning = ...
    end = ...
    while ...:
        middle = ...
        if L[middle] == target:
            return True
        elif L[middle] > target:
            ...
        else:
            ...
    return False
binary_search tests should include

• empty list
• list of length 1: target in list and not in list
• small list, both even and odd lengths
• larger list
  – target “outside” list, i.e. target < L[0] or target > L[len(L)-1]
  – target in the list, various positions (first, last, middle)
  – target not in the list, value between two list consecutive values
Worst Case running time of `binary_search`

- What is the runtime of each iteration?
- How many iterations are required at most?

Suppose $n = 2^k$:

- First comparison reduces search region size to $2^{k-1}$
- Second comparison reduces region size to $2^{k-2}$
- Third comparison reduces region size to $2^{k-3}$
- ...
- $m^{th}$ comparison reduces region size to $2^{k-m}$
- When is the region reduced to size 1?
Comments on running time for binary_search

- Worst case running time is $O(\log n)$
  - For $n \sim 1000$, will consider at most 11 elements ($2^{10} = 1024$)
  - For $n \sim 100,000$, will consider at most 18 elements ($2^{17} = 131072$)
  - For $n \sim 22,000,000$, will consider at most 26 elements ($2^{25} = 33,554,432$)
  - Doubling the size of list requires 1 more comparison worst-case!!!!
Comments and Questions on running time for `binary_search`

• Could be modified to return something other than a Boolean:
  – *What would be a good value?*

• Could be written recursively instead in Python and still have worst case run-time of $O(\log n)$
  – *Would the worst case for a recursive implementation in Racket still be $O(\log n)$?*
def sort_list(L):
    '''sorts L into increasing order
    Effects: L is mutated
    sort_list: (listof Int) -> None
    requires: No duplicate values in L

    Example: Suppose lst = [1,4,3,2],
    calling sort_list(lst) => None,
    but reorders lst as [1,2,3,4]
    '''
Sorting Algorithms

There are many different approaches to sorting. We will study the following algorithms and their runtimes:

• Selection sort
• Insertion sort
• Mergesort
Selection sort: basic idea

- Place the smallest entry into L[0]
- Place the second smallest entry into L[1]
- Place the third smallest entry into L[2]
- ...
- After step n-1, the list is sorted.
def selection_sort(L):
    n = len(L)
    positions = list(range(n-1))
    for i in positions:
        min_pos = i
        for j in range(i,n):
            if L[j] < L[min_pos]:
                min_pos = j
        temp = L[i]
        L[i] = L[min_pos]
        L[min_pos] = temp
Selection sort: Runtime

- Before the loop: $O(n)$
- Inner loop: $O(n)$ each iteration
- Outer loop: $O(n)$ iterations

$\rightarrow O(n) + O(n) \times O(n)$
$\rightarrow O(n^2)$

Selection sort is, perhaps, the easiest sorting algorithm, but there are faster algorithms.

Next up: Insertion sort
Insertion Sort: an introduction

• Idea: consider the list to be in two pieces

  SORTED  UNSORTED

  SORTED  UNSORTED

• Inserting the first item in "Unsorted" into its proper place in "Sorted", shrinks "Unsorted" and enlarges "Sorted"

• Repeat this process until "Unsorted" is empty
Insertion sort: an example
Sorting L=[5,8,2,4,3,1,9,6]

- [5] is sorted, insert 8
- [5,8] is sorted, insert 2
- [2,5,8] is sorted, insert 4
- [2,4,5,8] is sorted, insert 3
- [2,3,4,5,8] is sorted, insert 1
- [1,2,3,4,5,8] is sorted, insert 9
- [1,2,3,4,5,8,9] is sorted, insert 6
- [1,2,3,4,5,6,8,9] is sorted. No more to insert.
def insert(L, pos):
    '''sorts L[0:pos]
    insert: (listof Int) Nat -> None
    requires: L[0:pos-1] is sorted.''
    while pos > 0 and L[pos] < L[pos-1]:
        temp = L[pos]
        L[pos] = L[pos-1]
        L[pos-1] = temp
        pos = pos-1

def insert_sort(L):
    for i in range(1,len(L)):
        insert(L,i)
Running time of `insert_sort`

`insert_sort` requires $O(n)$ calls to `insert`

`insert` requires at most $O(n)$ while loop iterations

Each while loop body requires $O(1)$ steps

$\Rightarrow O(n) \ast O(n) \ast O(1)$

$\Rightarrow O(n^2)$

*What is the best-case?*
Mergesort—another sorting algorithm

Consider the following approach

– Divide the list into two halves
– Sort the first half
– Sort the second half
– Combine the sorted lists together

⇒ Done!

*Mergesort is a "Divide and Conquer" algorithm.*
Mergesort questions

• How to split the list?
  – Find the middle – before and after
• How to sort smaller lists?
  – Use same idea again (mergesort recursively)
• When to stop recursion?
  – When the list is empty
• How to combine the parts?
  – merge
Merge helper function

Suppose $L_1$, $L_2$ are in increasing order. To merge:

- If $L_1$ is empty, the merged list is $L_2$
- If $L_2$ is empty, the merged list is $L_1$
- If $L_1[0] < L_2[0]$, then the merged list is:
  $$[L_1[0]] + \text{merge}(L_1[1:], L_2)$$
- Otherwise, the merged list is:
  $$[L_2[0]] + \text{merge}(L_1, L_2[1:])$$

Note: we will use a modified version of this algorithm to get a better run-time
def merge(L1,L2,L):
    pos1,pos2,posL = 0,0,0
    while (pos1 < len(L1)) and (pos2 < len(L2)):
        if L1[pos1] < L2[pos2]:
            L[posL] = L1[pos1]
            pos1 += 1
        else:
            L[posL] = L2[pos2]
            pos2 += 1
        posL += 1
    while (pos1 < len(L1)):
        L[posL] = L1[pos1]
        pos1, posL = pos1+1, posL+1
    while (pos2 < len(L2)):
        L[posL] = L2[pos2]
        pos2, posL = pos2+1, posL+1

Note: L1 and L2 must be sorted before merge is called, and L is combined length of L1 and L2

pos1, pos2, posL are list positions
Running Time of $\text{merge}$

• Suppose $\text{len}(L_1) = m$ and $\text{len}(L_2) = p$

• Maximum number of while loop iterations is $O(m + p)$

• Each loop is $O(1)$

• Total $\rightarrow O(m + p)$

• Note: if $m = n/2$ and $p = n/2$, then $O(n)$
def mergesort(L):
    if len(L) < 2: return
    mid = len(L)//2
    L1 = L[:mid]
    L2 = L[mid:]
    mergesort(L1)
    mergesort(L2)
    merge(L1,L2,L)

Running time:

\[
T(n) = O(n) + 2T\left(\frac{n}{2}\right) \rightarrow O(n \log n)
\]
Calls to `mergesort`

```
    n
   / | \
  n/2 n/2
 /   /   \
 n/4 n/4 n/4 n/4
```

1 1 1 1 1 1 1 1 1
Running time of *mergesort*

- Consider the time across each level of the tree.
  - How long does it take to divide the lists in half?
  - How long does it take to merge the lists together?
- How many levels of the tree are there?
- Total running time is $O(n \log n)$
Built-in sorted and sort

• Python: `sorted`
  – Built-in function
  – Consumes a list and returns a sorted copy

• Python: `sort`
  – A list method
  – Consumes a list and modifies into sorted order

• Additional arguments can be provided to change the sort (e.g. into decreasing order)

• $O(n \log n)$ runtime for Python's sorting functions
Goals of Module 08

• Understand how linear and binary search work
• Be able to compare running times of searching algorithms
• Understand how insertion sort, selection sort and mergesort work
• Be able to compare running times of sorting algorithms