Module 07: Efficiency

Topics:

• Basic introduction to run-time efficiency
• Analyzing recursive code
• Analyzing iterative code
Consider the following two ways to calculate the maximum of a nonempty list. Why is one so much slower than the other?

def list_max(values):
    max_so_far = values[0]
    for v in values:
        if v > max_so_far:
            max_so_far = v
    return max_so_far

def list_max1(x):
    if len(x) == 1:
        return x[0]
    elif x[0] > list_max1(x[1:]):
        return x[0]
    else:
        return list_max1(x[1:])
Comparing Algorithms

Suppose you have two algorithms to solve a problem. How can we determine which one is better?


• We will use efficiency to compare algorithms.
Efficiency

The most common measure of efficiency is *time efficiency*, or how long it takes an algorithm to solve a problem.

- Depends on its implementation

Another measure of efficiency is *space efficiency*, or how much space (memory) an algorithm requires to solve a problem.
Efficiency: measurement of Running Time of an algorithm

What is our unit of measurement? Seconds?
• Dependent on when statement made, what computer, how much RAM, what language used, what OS, etc.
• Do we consider the average time over all possible problems? Just one? Which one?

The actual time taken is not a great choice. Instead, we will count number of steps or basic operations performed.
Example

What is the number of operations executed when calling this function?

```python
def sum_all(values):
    sum = 0
    ind = 0
    upper = len(values)
    while (ind < upper):
        sum = sum + values[ind]
        ind = ind + 1
    return sum
```
Input size

• Let $n$ refer to the size of the problem
  – Length of list
  – Number of characters in a string
  – Number of digits in a Nat
  – Meaning should be specified if not clear

• Running time is always stated as a function of $n$. We denote it by $T(n)$
Running time depends on data values, not just input size

- Assume $n = \text{len}(L)$
- How many steps are taken by the following code?

```python
ind = 0
length = \text{len}(L)
while (ind < length) and (L[ind] > 0):
    ind = ind + 1
```
Terminology

• We will be pessimistic, and determine the largest value of $T(n)$ possible for a fixed $n$
  – Worst case running time
  – This is our default meaning of "run time"

• Sometimes we are interested in the best case, i.e. the minimum value of $T(n)$ possible for a fixed value of $n$
Big O notation

• In practice, we are not concerned with the difference between the running times $6n + 6$ and $174n + 32$.

• We are interested in the order of a running time. The order is the dominant term without its coefficient.

• The dominant term in both $6n + 6$ and $174n + 32$ is $n$, so both are "Order n", denoted $O(n)$

• This is called the asymptotic run time
Big O Examples

• $2016 = O(1)$
• $12 \log n + 45 = O(\log n)$
• $12 \log n + 45n = O(n)$
• $20n \log n + 3n + 27 = O(n \log n)$
• $3 + n + n^2 + 2^n = O(2^n)$
Important Big O information

• In this course, we will encounter only a few orders (arranged from smallest to largest):
  \[ O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(2^n) \]

• Note that these relationships hold as \( n \rightarrow \infty \)

• When comparing algorithms, the most efficient algorithm is the one with the lowest order.

• If two algorithms have the same order, they are considered equivalent, but may not take exactly the same number of steps.
What is the running time of this code?

def list_max(values):
    max_so_far = values[0]
    for v in values:
        if v > max_so_far:
            max_so_far = v
    return max_so_far
Big O arithmetic

• When adding two orders, the result is the larger of the two orders.
  – $O(\log n) + O(n) = O(n)$
  – $O(1) + O(1) = O(1)$

• How can we use this result?
  – Break code into blocks that run one after the other
  – If you determine the asymptotic run times of the blocks independently, then just add them to get the overall run time.
Algorithm analysis

- An important skill in Computer Science is the ability to analyze a program and determine the order of its running time.
- In this course, you will not need to count operations exactly.
- Our goal is to give you experience and to work towards building your intuition.

```python
sum = 0
for x in lst:
    sum = sum + x
```

Each item in the list is retrieved once, so running time is $O(n)$.
Basic Operations in Python

We will make the following assumptions

• Numerical operations:
  – +, -, *, /, = are $O(1)$
  – max(a, b), min(a, b) are $O(1)$

• String operations, where $n = \text{len}(s)$
  – len(s), s[k] are $O(1)$
  – s + t is $O(n + \text{len}(t))$
  – Most string methods (e.g. count, find, lower) are $O(n)$

• print and input are dependent on the length of what is being printed and read in
Basic List Operations, where \( n = \text{len}(L) \)

We will make the following **assumptions**:

- \( \text{len}(L) \), \( L[k] \) are \( O(1) \)
- \( L + M \) is \( O(n + \text{len}(M)) \)
  - \( L + [x] \) is \( O(n) \)
- \( \text{sum}(L) \), \( \text{max}(L) \), \( \text{min}(L) \) are \( O(n) \)
- \( L[a:b] \) is \( O(b - a) \), so at most \( O(n) \)
  - \( L[1:] \) is \( O(n) \)
- \( L.append(x) \) is \( O(1) \)
More basic list operations, where $n = \text{len}(L)$

We will make the following **assumptions**:

- `list(range(n))` is $O(n)$
- Most other list methods on `L` (e.g. `count`, `index`, `insert`, `pop`, `remove`) are $O(n)$
- `L.sort()` is $O(n \log n)$
Here are two ways to duplicate a list. Which is most efficient?

def duplicate1(L):
    extra = []
    for x in L:
        extra.append(x)
    return extra

def duplicate2(L):
    extra = []
    for x in L:
        extra = extra + [x]
    return extra
General Procedure for analyzing a loop

• Determine the number of iterations
• For each iteration, determine the running time of body of the loop
  – Each loop body may have the same running time, but that is not guaranteed
• Add together the running time of each loop body to get the overall running time
More Big O arithmetic

• When multiplying two orders, the result is the product of the two orders.
  – \( O(\log n) \times O(n) = O(n \log n) \)
  – \( O(n) \times O(n) = O(n^2) \)

• How can we use this result?
  – Determine the asymptotic run time of the number of iterations of a loop
  – Determine the asymptotic run time of the body of the loop
  – Multiply them to get the overall asymptotic run time
Warning: The following code fragments do NOT have the same runtime. Why?

```
diff = 0
for x in L:
    diff += abs(x - sum(L)/len(L))
```

```
diff = 0
mean = sum(L)/len(L)
for x in L:
    diff += abs(x-mean)
```

Be very careful about what steps are put inside the loop body. Try to move non-\(O(1)\) steps outside the loop body, when possible.
What if there are nested loops?

• You can take different approaches:
  – Work from the innermost loop to the outermost
  – Work from the outermost loop to the innermost

• Nested loops can lead to nested sums
What is the running time of `mult_table(n)`?

def mult_table(n):
    table = [0]*n
    row = 0
    columns = list(range(n))
    while row < n:
        this_row = []
        for c in columns:
            this_row.append((row+1)*(c+1))
        table[row] = this_row
        row = row + 1
    return table
Useful summations

- $\sum_{i=1}^{n} 1 = O(n)$
- $\sum_{i=1}^{n} i = O(n^2)$
- $\sum_{i=1}^{n} n = O(n^2)$
- $\sum_{i=1}^{n} \sum_{j=1}^{n} 1 = O(n^2)$
How do we determine runtime of recursive code?

```python
def list_max(L):
    if len(L) == 1:
        return L[0]
    else:
        return max(L[0],
                  list_max(L[1:]))
```

- Count steps for:
  - Determine `len(L)`
  - Compare to 1
  - Calculate `L[0]`
  - Calculate `L[1:]`
  - Call `list_max` recursively on a list of length n-1
  - Determine `max` of two values

- $T(n) = O(n) + T(n-1)$
More generally ...

• To help in analyzing recursive code, we will use basic recurrence relations.

• We will express the running time of a problem of size \( n \) in terms of
  – Running time of the code other than recursion
  – Running time of recursive call(s)

• For example:
  \[ T(n) = O(n) + T(n - 1) \]
Helpful recurrence relations

• Once we have such a recurrence relation, use the following rules to determine the overall running time.
  
• \( T(n) = O(1) + T(n - 1) \rightarrow O(n) \)

• \( T(n) = O(n) + T(n - 1) \rightarrow O(n^2) \)

• \( T(n) = O(1) + T(n/2) \rightarrow O(\log n) \)

• \( T(n) = O(n) + 2T(n/2) \rightarrow O(n \log n) \)
  
  \(- T(n) = O(n) + T(n/2) \rightarrow O(n) \)

• \( T(n) = O(1) + T(n - 1) + T(n - 2) \rightarrow O(2^n) \)
  
  \(- T(n) = O(1) + 2T(n - 1) \rightarrow O(2^n) \)

  \(- T(n) = O(n) + T(n - 1) + T(n - 2) \rightarrow O(2^n) \)

  \(- T(n) = O(n) + 2T(n - 1) \rightarrow O(2^n) \)
Here are two ways to find maximum in a list. Which is more efficient?

```python
def list_max1(x):
    if len(x) == 1:
        return x[0]
    elif x[0] > list_max1(x[1:]):
        return x[0]
    else:
        return list_max1(x[1:])

def remember_max(m, y):
    if len(y) == 0:
        return m
    elif m > y[0]:
        return remember_max(m, y[1:])
    else:
        return remember_max(y[0], y[1:])

def list_max2(x):
    return remember_max(x[0], x[1:])
```
Analysing abstract list functions

• $\text{map}(f,L), \text{filter}(f,L)$ are at least $O(n)$

• Actual running time depends on running time of $f$

• Hint: *Analyse the program as if it were a loop instead of* $\text{map}$ *or* $\text{filter}$
Determine the running times

```python
def duplicate3(L):
    return list(map(lambda x: x, L))

def first_chars(words):
    return list(map(lambda t: t[0],
                    filter(lambda s: len(s) > 0, words)))

def list_of_lists(n):
    return list(map(lambda x:
                    list(range(n)),
                    range(n)))
```
Overall comments

• We've provided just a basic introduction to runtime analysis
  – Especially for recursive code
  – We have made some simplifications
• The topic is very important, though, and even an introduction can help you design better programs.
• Like this topic?
  – CS234 (non-majors)
  – CS240 (majors)
# Summary of Common Runtimes

<table>
<thead>
<tr>
<th>Common Name</th>
<th>O(1)</th>
<th>O(log n)</th>
<th>O(n)</th>
<th>O(n log n)</th>
<th>O(n^2)</th>
<th>O(2^n)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Constant</td>
<td>Logarithmic</td>
<td>Linear</td>
<td>Log Linear</td>
<td>Quadratic</td>
<td>Exponential</td>
</tr>
</tbody>
</table>

Functions grow more quickly

Algorithms run more slowly
Goals of Module 07

• Understand how to analyze Python code to determine its running time, including
  – Recursion
  – Iteration
  – Abstract list functions

• Understand basic run time categories