Module 08: Searching and Sorting Algorithms

Topics:

• Searching algorithms
• Sorting algorithms
Application: Searching a list

Suppose you have a list $L$. How could you determine if a particular value is in that list, if $L$ is in no particular order?

Algorithm (called Linear Search)

• Check the first element in $L$: is it the one?
  – If Yes, return True
  – Else, check the next value
• The value is not in the list if you don’t find it (return False)
Implementing Linear Search

def linear_search (L, target):
    '''returns True if target is in L,
    False otherwise
    linear_search:(listof X) X -> Bool
    Note: equivalent to: target in L
    '''
    for val in L:
        if val == target:
            return True
    return False
Running Time of `linear_search`

• Let `n = len(L)`

• Best Case:
  – If target is in first position, we find it right away $\Rightarrow O(1)$

• Worst Case:
  – If target is not in $L$, we have to check all $n$ elements $\Rightarrow O(n)$
  – What is the other worst case?
Alternatives to Linear Search

• If $L$ is unsorted, we can’t do any better than Linear Search.

• How could we improve Linear Search if $L$ was sorted into increasing order?
  – Are there situations in which we could stop earlier?
  – Is this any faster in the worst case?
A better approach: Binary Search

- Suppose $L$ is a listing of the taxpayers in Canada, sorted into increasing order by Social Insurance numbers.
- Approximately 22,000,000 entries
- Look at $L[11000000]$
- Is it the target taxpayer?
  - If yes, stop.
  - If not, is $target < L[11000000]$?
    - If yes, then $target$ is in the first half of $L$
    - If not, then $target$ is in the second half of $L$
  - Repeat this process for the half containing $target$
Developing `binary_search`

- We need to determine how to keep track of the section of the list still being searched
  - Variables `beginning, end`
  - Determine their initial values
- Determine the `middle` position
- If \( L[middle] \) is target, return `stop`
- Otherwise, update `beginning` and `end`
- Determine when we to continue (or stop) searching
Starting the implementation

def binary_search(L, target):
    beginning = ...
    end = ...
    while ...:
        middle = ...
        if L[middle] == target:
            return True
        elif L[middle] > target:
            ...
        else:
            ...
    return False
binary_search tests should include

• empty list
• list of length 1: target in list and not in list
• small list, both even and odd lengths
• larger list
  – target “outside” list, i.e. target < L[0] or target > L[len(L) - 1]
  – target in the list, various positions (first, last, middle)
  – target not in the list, value between two list consecutive values
Worst Case running time of \texttt{binary\_search}

- What is the runtime of each iteration?
- How many iterations are required at most?

Suppose $n = 2^k$:

- First comparison reduces search region size to $2^{k-1}$
- Second comparison reduces region size to $2^{k-2}$
- Third comparison reduces region size to $2^{k-3}$
- ...  
- $m^{th}$ comparison reduces region size to $2^{k-m}$
- When is the region reduced to size 1?
Comments on running time for binary_search

• Worst case running time is $O(\log n)$
  – For $n \sim 1000$, will consider at most 11 elements ($2^{10} = 1024$)
  – For $n \sim 100,000$, will consider at most 18 elements ($2^{17} = 131072$)
  – For $n \sim 22,000,000$, will consider at most 26 elements ($2^{25} = 33,554,432$)
  – Doubling the size of list requires 1 more comparison worst-case!!!!
Comments and Questions on running time for `binary_search`

- Could be modified to return something other than a Boolean:
  - *What would be a good value?*

- Could be written recursively instead in Python and still have worst case run-time of $O(\log n)$
  - *Would the worst case for a recursive implementation in Racket still be $O(\log n)$?*
def sort_list(L):
    '''sorts L into increasing order
    Effects: L is mutated
    sort_list: (listof Int) -> None
    requires: No duplicate values in L
    Example: Suppose lst = [1,4,3,2],
    calling sort_list(lst) => None,
    but reorders lst as [1,2,3,4]
    '''
Sorting Algorithms

There are many different approaches to sorting. We will study the following algorithms and their runtimes:

• Selection sort
• Insertion sort
• Mergesort
Selection sort: basic idea

• Place the smallest entry into L[0]
• Place the second smallest entry into L[1]
• Place the third smallest entry into L[2]
• ...
• After step n-1, the list is sorted.
Selection sort implementation

```python
def selection_sort(L):
    n = len(L)
    positions = list(range(n-1))
    for i in positions:
        min_pos = i
        for j in range(i, n):
            if L[j] < L[min_pos]:
                min_pos = j
        temp = L[i]
        L[i] = L[min_pos]
        L[min_pos] = temp
```

Selection sort: Runtime

- Before the loop: $O(n)$
- Inner loop: $O(n)$ each iteration
- Outer loop: $O(n)$ iterations

$\Rightarrow O(n) + O(n) \times O(n)$

$\Rightarrow O(n^2)$

Selection sort is, perhaps, the easiest sorting algorithm, but there are faster algorithms.

Next up: Insertion sort
Insertion Sort: an introduction

• Idea: consider the list to be in two pieces

  SORTED    UNSORTED

• Inserting the first item in "Unsorted" into its proper place in "Sorted", shrinks "Unsorted" and enlarges "Sorted"

• Repeat this process until "Unsorted" is empty
Insertion sort: an example
Sorting L=[5,8,2,4,3,1,9,6]

• [5] is sorted, insert 8
• [5,8] is sorted, insert 2
• [2,5,8] is sorted, insert 4
• [2,4,5,8] is sorted, insert 3
• [2,3,4,5,8] is sorted, insert 1
• [1,2,3,4,5,8] is sorted, insert 9
• [1,2,3,4,5,8,9] is sorted, insert 6
• [1,2,3,4,5,6,8,9] is sorted. No more to insert.
def insert(L, pos):
    '''sorts L[0:pos]
    insert: (listof Int) Nat -> None
    requires: L[0:pos-1] is sorted.'''
    while pos > 0 and L[pos] < L[pos-1]:
        temp = L[pos]
        L[pos] = L[pos-1]
        L[pos-1] = temp
        pos = pos-1

def insert_sort(L):
    for i in range(1,len(L)):
        insert(L,i)
Running time of `insert_sort`

`insert_sort` requires $O(n)$ calls to `insert`

`insert` requires at most $O(n)$ while loop iterations

Each while loop body requires $O(1)$ steps

$\Rightarrow O(n) \times O(n) \times O(1)$

$\Rightarrow O(n^2)$

What is the best-case?
Mergesort— another sorting algorithm

Consider the following approach

– Divide the list into two halves
– Sort the first half
– Sort the second half
– Combine the sorted lists together

⇒ Done!

*Mergesort is a "Divide and Conquer" algorithm.*
Mergesort questions

• How to split the list?
  – Find the middle – before and after

• How to sort smaller lists?
  – Use same idea again (mergesort recursively)

• When to stop recursion?
  – When the list is empty

• How to combine the parts?
  – merge
Merge helper function

Suppose \( L_1, L_2 \) are in increasing order. To merge:

• If \( L_1 \) is empty, the merged list is \( L_2 \)
• If \( L_2 \) is empty, the merged list is \( L_1 \)
• If \( L_1[0] < L_2[0] \), then the merged list is:
  \[
  [L_1[0]] + \text{merge}(L_1[1:], L_2)
  \]
• Otherwise, the merged list is:
  \[
  [L_2[0]] + \text{merge}(L_1, L_2[1:])
  \]

Note: we will use a modified version of this algorithm to get a better run-time
def merge(L1,L2,L):
    pos1,pos2,posL = 0,0,0
    while (pos1 < len(L1)) and (pos2 < len(L2)):
        if L1[pos1] < L2[pos2]:
            L[posL] = L1[pos1]
            pos1 += 1
        else:
            L[posL] = L2[pos2]
            pos2 += 1
        posL += 1
    while (pos1 < len(L1)):
        L[posL] = L1[pos1]
        pos1, posL = pos1+1, posL+1
    while (pos2 < len(L2)):
        L[posL] = L2[pos2]
        pos2, posL = pos2+1, posL+1

Note: \( L_1 \) and \( L_2 \) must be sorted before \texttt{merge} is called, and \( L \) is combined length of \( L_1 \) and \( L_2 \).

\( \text{pos1, pos2, posL} \) are list positions
Running Time of \texttt{merge}

• Suppose \texttt{len}(L1) = m and \texttt{len}(L2) = p

• Maximum number of while loop iterations is \(O(m + p)\)
• Each loop is \(O(1)\)
• Total \(\rightarrow O(m + p)\)

• Note: if \(m = n/2\) and \(p = n/2\), then \(O(n)\)
def mergesort(L):
    if len(L) < 2: return
    mid = len(L)//2
    L1 = L[:mid]
    L2 = L[mid:]
    mergesort(L1)
    mergesort(L2)
    merge(L1,L2,L)

Running time:

\[ T(n) = O(n) + 2T\left(\frac{n}{2}\right) \rightarrow O(n \log n) \]
### Sorting Run-time Summary

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Best Case</th>
<th>Worst Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>Selection Sort</td>
<td>$O(n^2)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>Linear Insertion Sort</td>
<td>$O(n)$</td>
<td>$O(n^2)$</td>
</tr>
<tr>
<td>Mergesort</td>
<td>$O(n \log n)$</td>
<td>$O(n \log n)$</td>
</tr>
</tbody>
</table>
Built-in `sorted` and `sort`

- **Python: `sorted`**
  - Built-in function
  - Consumes a list and returns a sorted copy
- **Python: `sort`**
  - A list method
  - Consumes a list and modifies into sorted order
- Additional arguments can be provided to change the sort (e.g. into decreasing order)
- $O(n \log n)$ runtime for Python's sorting functions
Examples of `sort` and `sorted`

\[ L = \begin{bmatrix} [3,4], &[5,1], &[0,0], &[2,8], &[5,0] \end{bmatrix} \]

\[ N1 = \text{sorted}(L, \text{key} = \lambda x: x[0]+x[1]) \]

\[ N2 = \text{sorted}(L, \text{key} = \lambda x: x[1], \text{reverse=True}) \]

\[ L = \begin{bmatrix} "cs", &"116", &"ls", &"cool" \end{bmatrix} \]

\[ L.\text{sort}(\text{key} = \lambda s: \text{len}(s)) \]

\[ L.\text{sort}(\text{key} = \lambda s: s.\text{count}(\text{'}1\text{'}), \text{reverse=True}) \]
Goals of Module 08

• Understand how linear and binary search work
• Be able to compare running times of searching algorithms
• Understand how insertion sort, selection sort and mergesort work
• Be able to compare running times of sorting algorithms