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WARNING: Slides do not include all class material; if you have missed a lecture, make sure to find out from a classmate what material was presented verbally or on the board.
Case study

Problem: When colour is applied to a part of a web page, what other parts of the page will obtain the same colour?
For example, if a section is coloured, the paragraphs and lists will get the same colour.

Recipe for user/plan

1. Determine types of data and operations.
2. For each type, choose/modify/create an ADT.
3. Develop pseudocode algorithm using ADT operations.
4. Calculate cost of algorithm with respect to costs of operations.
5. Using information from provider, choose best option.
Representing a web page

The Document Object Model represents a web page as a tree.
Tree review

Which of these trees are the same?
Basic definitions

A **tree** is formed of **nodes** connected by **edges**. (This is not the same as a node in a linked list.)

In a **rooted tree**, one node is designated as the **root** of the tree.

In a drawing where the root is at the top, an edge connects a **parent** to a **child**, where the parent is the node closer to the root.

Nodes that share a parent are **siblings**.

A node without children is a **leaf**; a node that is not a leaf is an **internal node**.

A node’s parent, its parent’s parent, and so on up to the root are its **ancestors**; a node’s children, children’s children, and so on are its **descendants**.

A node and all its descendants form the **subtree rooted at** that node.
Types of rooted trees

A tree is **unordered** if there is no order specified on the children of a node, and **ordered** otherwise.

A **binary tree** is a tree in which each parent has at most two children and each child is specified as either a **left child** or a **right child**. In a binary tree, the subtree rooted at the left child is the **left subtree** and the subtree rooted at the right child is the **right subtree**.
Terminology for rooted trees

The **path** between nodes $n_0$ and $n_k$ is the sequence of nodes $\{n_0, n_1, \ldots, n_k\}$ such that there is an edge between $n_i$ and $n_{i+1}$ for all $0 \leq i < k$; a path is **simple** if each node appears at most once in the sequence. The **length** of a path is the number of edges in the path.

The **depth of a node** $n$ is the length of the path between $n$ and the root; a root is thus at depth 0. All nodes of the same depth are on the same **level**.

The **height of a node** $n$ is the maximum length of any path between $n$ and a leaf in the subtree rooted at $n$; a leaf thus has height 0. The **height of a tree** is the height of the root of the tree.
Nodes as positions to store data

In earlier ADTs, we accessed data by position, such as:

- top (ADT Stack)
- front (ADT Queue)
- index (ADT Indexed Sequence)
- rank (ADT Ranking)
- row and column (ADT Grid)

We can navigate in a tree by starting at the root, choosing a child, and so on to find a specific node.

How do we refer to a particular node?

Instead of using the path from the root, we’ll associate a unique ID with each node. The type of data used for the ID may depend on the data structure implementing the ADT (e.g. index in an array or pointer to a node in a linked structure).
Data stored in nodes

Depending on the application, a node of a tree can store various types of data, such as:

- a value
- a weight
- a colour

For now, we will define our ADTs such that each node stores a single value.
Search operations for trees

- Find the value of a node
- Find the root of the tree
- Find the parent of a node
- Find a specific child of a node
- Find all children of a node
- Find the node storing a particular value
- Find all nodes storing a particular value
- Find all nodes in the tree

Issues to consider:

How can multiple nodes be returned?

Use a Group B ADT that allows us to extract all the data items, possibly in some specific order.
Modification operations for trees

- Add a new node
- Delete a node
- Delete a subtree
- Change the value stored in a node
- Swap values stored in two nodes
- Swap subtrees

Issues to consider:

What remains after a node is deleted?
Initially just delete leaves.
Initially start with binary trees.
ADT Binary Tree, without modifications

Preconditions: For all $B$ is a binary tree and $Node$ is a node in $B$; for $Root B$ is not empty.

<table>
<thead>
<tr>
<th>Name</th>
<th>Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$CREATE()$</td>
<td>a new empty binary tree</td>
</tr>
<tr>
<td>$IS_EMPTY(B)$</td>
<td>$True$ if empty, else $False$</td>
</tr>
<tr>
<td>$ROOT(B)$</td>
<td>root of $B$</td>
</tr>
<tr>
<td>$VALUE(B, Node)$</td>
<td>value stored in $Node$</td>
</tr>
<tr>
<td>$PARENT(B, Node)$</td>
<td>parent of $Node$ if any, else $False$</td>
</tr>
<tr>
<td>$LEFT_CHILD(B, Node)$</td>
<td>left child of $Node$ if any, else $False$</td>
</tr>
<tr>
<td>$RIGHT_CHILD(B, Node)$</td>
<td>right child of $Node$ if any, else $False$</td>
</tr>
</tbody>
</table>
ADT Binary Tree, modifications

Preconditions: For all $B$ is a binary tree, $Node$ is a node in $B$, and $Data$ is a data item; for $ADD\_LEAF$ either $Par$ and $Side$ are both empty or $Par$ is a node in $B$ and $Side$ is $Left$ or $Right$; for $DELETE\_LEAF$ $Node$ is a leaf.

Postconditions: Mutation by $SET\_VALUE$ (sets value of $Node$ to $Data$), $ADD\_LEAF$ (creates a new node containing $Data$ to replace/add the root if $Par$ is empty and to replace/add the $Side$ subtree of $Par$ otherwise), and $DELETE\_LEAF$ (deletes $Node$).

<table>
<thead>
<tr>
<th>Name</th>
<th>Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SET_VALUE(B, Node, Data)$</td>
<td></td>
</tr>
<tr>
<td>$ADD_LEAF(B, Par, Side, Data)$</td>
<td>new added node storing $Data$</td>
</tr>
<tr>
<td>$DELETE_LEAF(B, Node)$</td>
<td></td>
</tr>
</tbody>
</table>
Example of use of ADT Binary Tree operations

\[
\begin{align*}
&\text{Fruit} \leftarrow \text{CREATE()} \\
&\text{Apple} \leftarrow \text{ADD\_LEAF(Fruit, None, None, apple)} \\
&\text{Guava} \leftarrow \text{ADD\_LEAF(Fruit, Apple, Left, guava)} \\
&\text{Peach} \leftarrow \text{ADD\_LEAF(Fruit, Apple, Right, peach)} \\
&\text{Mango} \leftarrow \text{ADD\_LEAF(Fruit, Guava, Right, mango)} \\
&\text{One} \leftarrow \text{ROOT(Fruit)} \\
&\text{Two} \leftarrow \text{PARENT(Fruit, Mango)} \\
&\text{Three} \leftarrow \text{LEFT\_CHILD(Fruit, Guava)} \\
&\text{Four} \leftarrow \text{RIGHT\_CHILD(Fruit, Guava)} \\
&\text{DELETE\_LEAF(Fruit, Peach)} \\
&\text{Five} \leftarrow \text{RIGHT\_CHILD(Fruit, Apple)}
\end{align*}
\]
Linked implementation of ADT Binary Tree

Data structures:

- Variable pointing to root node, if any
- Nodes storing data items and three pointers *Parent* (to parent), *Left* (to left child), and *Right* (to right child)

Worst-case running times of operations are all in $\Theta(1)$. Cost of searching for a node from the root depends on depth.

Caution: The word “node” can mean either or both of “node in a tree” and “node in a linked implementation.”
Contiguous implementation of ADT Binary Tree

Observations:

- For node at index $p$, index of left child is $2p + 1$
- For node at index $p$, index of right child is $2p + 2$
- For node at index $p$, index of parent is $\lfloor (p - 1)/2 \rfloor$
Exploring a contiguous implementation

Observations:

- For node at index $p$, index of left child may not be $2p + 1$
- For node at index $p$, index of right child may not be $2p + 2$
- For node at index $p$, index of parent may not be $\lfloor (p - 1)/2 \rfloor$
More terminology for binary trees

In a perfect binary tree, each node has zero or two children and all leaves are at the same depth.

In a complete binary tree every level, except possibly the last, is completely filled, and all nodes on the last level are as far to the left as possible.
Contiguous implementation of a ADT Binary Tree

Data structures:

- Array storing values level by level as if all nodes were present
- Variable $Last$ with the last index storing an element
Computing a sibling

Options for computing a sibling:
- Write an algorithm using existing ADT operations.
- Augment the ADT by adding a new operation.

Use existing ADT operations:
- Use \texttt{PARENT} to find parent.
- Use \texttt{LEFT\_CHILD} and \texttt{RIGHT\_CHILD} to find children of parent.
- If there is only one, return \textit{False}.
- If there are two, return the one which is not the node itself.
Modifying the implementations for the augmented ADT

Linked implementation:
- Use *Parent* pointer to find parent.
- Use *Left* and *Right* pointers to find children of parent.
- If there is only one, return *False*.
- If there are two, return the one which is not the node itself.
- Cost is $\Theta(1)$.

Contiguous implementation:
- For node at odd index $p$, index of sibling is $p + 1$ (if $p + 1$ is at most *Last*).
- For node at even positive index $p$, index of sibling is $p - 1$.
- Cost is $\Theta(1)$. 
ADT Ordered Tree

Preconditions: For all $O$ is an ordered tree, $Node$ is a node in $O$, and $Data$ is a data item; for $\text{ONE\_CHILD Index}$ is a nonnegative integer at most one less than the number of children of $Node$; for $\text{ADD\_LEAF Par}$ is a node in $O$ or empty and $Sib$ is a child of node $Par$ or empty.

Postconditions: Mutation by $\text{ADD\_LEAF}$ (creates a new node containing $Data$ to replace/add the root if $Par$ is empty, as the first child of $Par$ if $Sib$ is empty, and otherwise as the next sibling of $Sib$).

($\text{CREATE, IS\_EMPTY, ROOT, VALUE, PARENT, SET\_VALUE, and DELETE\_LEAF}$ like in ADT Binary Tree)

<table>
<thead>
<tr>
<th>Name</th>
<th>Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{CHILDREN}(O, Node)$</td>
<td>all children of $Node$ (Group B ADT)</td>
</tr>
<tr>
<td>$\text{ONE_CHILD}(O, Node, Index)$</td>
<td>child $Index$</td>
</tr>
<tr>
<td>$\text{ADD_LEAF}(O, Par, Sib, Data)$</td>
<td>new added node storing $Data$</td>
</tr>
</tbody>
</table>
Linked implementation of ADT Ordered Tree

Data structures:

- Variable *Root* pointing to root node, if any
- Nodes storing data items and three pointers *Prev* (to parent if first child or previous sibling otherwise), *First* (to first child), and *Next* (to next sibling)
Pseudocode for \( \text{PARENT}(O, \text{Node}) \)

Use dot notation for fields inside a node in the linked structure.

```
if Root(O) == Node
    return False

Found ← False
Current ← Node

while not Found
    Previous ← Current.Prev
    if Current == Previous.First
        Found ← True
    else
        Current = Previous

return Previous
```
Computing the next sibling

Options for computing the next sibling:

- Write an algorithm using existing ADT operations.
- Augment the ADT by adding a new operation.

Using existing ADT operations:

- Use $PARENT$ to find parent.
- Use $CHILDREN$ to find children.
- Scan children to determine next sibling.

Modifying the linked implementation:

- Use $Next$ pointer to find next sibling.
Defining and implementing ADT Unordered Tree

ADT definition:
- Similar to ADT Ordered Tree
- Specify only parent, not sibling, when adding a node

Data structures:
- Same data structure as for ADT Ordered Tree
- Adapt algorithms to exploit fact that order of children is not significant
Returning all nodes

Options for returning all nodes:

- Write an algorithm using existing ADT operations.
- Augment the ADT by adding a new operation.

We can find all nodes by determining the root using $ROOT$ and then repeatedly using $CHILDREN$ (or $LEFT\_CHILD$ and $RIGHT\_CHILD$) to determine all descendants of the root.

Alternatively, we can use the recursive definition of a tree, in which each child of the root can be viewed as the root of a (smaller) tree. The result for the original tree will be determined using the results (obtained recursively) on the smaller trees.
Tree traversals

A **tree traversal** is an ordering of the nodes in the tree.

- In a **level order traversal**, nodes appear in increasing order of depth.
- In a **postorder traversal**, each node appears after its children.
- In a **preorder traversal**, each node appears before its children.
- In an **inorder traversal** (only in a binary tree), for each node all nodes in the left subtree come before the node and all nodes in the right subtree come after the node.

Note: Traversals can be viewed as templates for processing (not just numbering) nodes in a given order.
Traversal example

```
5
3
8 12 13 11
9 14 20 15 18
```

[Diagram of a tree with nodes labeled 3, 5, 10, 8, 12, 13, 11, 9, 14, 20, 15, 18]
Algorithms for traversals

Level order:

- Use \textit{ROOT} to set the current node to the root.
- Use \textit{CHILDREN} or \textit{LEFT\_CHILD} and \textit{RIGHT\_CHILD} to determine children of the current node.
- Add the children to an ADT Queue.
- Repeat the process with the first node in the queue as the current node.

All other traversals:

- Create a recursive algorithm that numbers nodes starting at a given number and produces the last number used.
- For postorder, number the subtrees (in order if a binary or ordered tree), and then give the next number to the root.
- For preorder, give the first number to the root and then number subtrees (in order if a binary or ordered tree).
- For inorder, number the left subtree, give the next number to the root, then number the right subtree.
Modifying an implementation

In a linked implementation, we can **thread** nodes together by adding an extra pointer from a node to the next node in the traversal.

In a contiguous implementation, we obtain a level-order traversal by examining values in order of increasing index.
Module summary

Topics covered:

- Case study: Web page
- Trees
- Decision tree
- Data stored in nodes
- Operations for trees
- ADT Binary Tree
- Linked implementation
- Contiguous implementation
- Perfect and complete trees
- Computing siblings
- ADT Ordered Tree
- Linked implementation
- Computing the next sibling
- ADT Unordered Tree
- Tree traversals