Module 2: Priority Queues

CS 240 - Data Structures and Data Management

Mark Petrick
Based on lecture notes by many previous cs240 instructors

David R. Cheriton School of Computer Science, University of Waterloo

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Abstract Data Types

**Abstract Data Type (ADT):** A description of information and a collection of operations on that information.

The information is accessed only through the operations.

We can have various realizations of an ADT, which specify:
- How the information is stored (*data structure*)
- How the operations are performed (*algorithms*)
Priority Queue ADT

**Priority Queue**: An ADT consisting of a collection of items (each having a *priority*) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing the item of *highest priority*

*deleteMax* is also called *extractMax*.

Applications: typical “todo” list, simulation systems

The above definition is for a *maximum-oriented* priority queue. A *minimum-oriented* priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin*. 
Realizations of Priority Queues

Attempt 1: Use unsorted arrays

- insert: $O(1)$
- deleteMax: $O(n)$

Using unsorted linked lists is identical.

Attempt 2: Use sorted arrays

- insert: $O(n)$
- deleteMax: $O(1)$

Using sorted linked-lists is identical.
Third Realization: Heaps

A *heap* is a certain type of binary tree.

Recall binary trees:
A binary tree is either
- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.
Heaps

A *max-heap* is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.

2. **Heap-order Property:** For any node $i$, key (priority) of parent of $i$ is larger than or equal to key of $i$.

A *min-heap* is the same, but with opposite order property.

**Lemma:** Height of a heap with $n$ nodes is $\Theta (\log n)$.
Insertion in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *bubble-up*:

```
bubble-up(v)
v: a node of the heap
1. while parent(v) exists and key(parent(v)) < key(v) do
2. swap v and parent(v)
3. v ← parent(v)
```

The new item bubbles up until it reaches its correct place in the heap.

Time: $O(\text{height of heap}) = O(\log n)$. 
deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a \textit{bubble-down}:

\begin{algorithm}
\caption{bubble-down($v$)}
\begin{algorithmic}[1]
\Statex $v$: a node of the heap
\Statex \textbf{1.} while $v$ is not a leaf do
\Statex \textbf{2.} $u \leftarrow$ child of $v$ with largest key
\Statex \textbf{3.} \textbf{if} key($u$) $>$ key($v$) \textbf{then}
\Statex \textbf{4.} \hspace{1em} swap $v$ and $u$
\Statex \textbf{5.} \hspace{1em} $v \leftarrow u$
\Statex \textbf{6.} \textbf{else}
\Statex \textbf{7.} \hspace{1em} break
\end{algorithmic}
\end{algorithm}

Time: $O$(height of heap) $= O(\log n)$. 
Priority Queue Realization Using Heaps

**heapInsert***(A, x)*

* A: an array-based heap, x: a new item
* 1. \( \text{size}(A) \leftarrow \text{size}(A) + 1 \)
* 2. \( A[\text{size}(A) - 1] \leftarrow x \)
* 3. **bubble-up**(A, \( \text{size}(A) - 1 \))

**heapDeleteMax**(A)

* A: an array-based heap
* 1. \( \text{max} \leftarrow A[0] \)
* 2. \( \text{swap}(A[0], A[\text{size}(A) - 1]) \)
* 3. \( \text{size}(A) \leftarrow \text{size}(A) - 1 \)
* 4. **bubble-down**(A, 0)
* 5. **return** max

Insert and deleteMax: \( O(\log n) \)
Building Heaps

**Problem statement:** Given \( n \) items (in \( A[0 \cdot \cdot n - 1] \)) build a heap containing all of them.

**Solution 1:** Start with an empty heap and insert items one at a time:

```
heapify1(A)
A: an array
1. initialize \( H \) as an empty heap
2. for \( i \leftarrow 0 \) to \( \text{size}(A) - 1 \) do
3.      heapInsert(\( H, A[i] \))
```

This corresponds to going from \( 0 \cdot \cdot n - 1 \) in \( A \) and doing *bubble-ups*
Worst-case running time: \( \Theta(n \log n) \).
Storing Heaps in Arrays

Let $H$ be a heap (binary tree) of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements level-by-level from top to bottom, in each level left-to-right.

It is easy to find parents and children using this array representation:

- the *left child* of $A[i]$ (if it exists) is $A[2i + 1]$,
- the *right child* of $A[i]$ (if it exists) is $A[2i + 2]$,
- the *parent* of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).
Problem statement: Given $n$ items (in $A[0 \cdots n - 1]$) build a heap containing all of them.

Solution 2: Using bubble-downs instead:

> heapify($A$)
> $A$: an array
> 1. $n \leftarrow \text{size}(A) - 1$
> 2. for $i \leftarrow \lfloor n/2 \rfloor$ downto 0 do
> 3. bubble-down($A$, $i$)

A careful analysis yields a worst-case complexity of $\Theta(n)$. A heap can be built in linear time.
Using a Priority Queue to Sort

$PQ - Sort(A)$
1. initialize $PQ$ to an empty priority queue
2. for $i \leftarrow 0$ to $n - 1$ do
3. $PQ.insert(A[i], A[i])$
4. for $i \leftarrow 0$ to $n - 1$ do
5. $A[n - 1 - i] \leftarrow PQ.deleteMax()$
HeapSort

\[
\text{HeapSort}(A)
\]
1. initialize \( H \) to an empty heap
2. \( \text{for } i \leftarrow 0 \text{ to } n - 1 \text{ do} \)
3. \( \text{heapInsert}(H, A[i]) \)
4. \( \text{for } i \leftarrow 0 \text{ to } n - 1 \text{ do} \)
5. \( A[n - 1 - i] \leftarrow \text{heapDeleteMax}(H) \)

HeapSort(A)
1. \( \text{heapify}(A) \)
2. \( \text{for } i \leftarrow 0 \text{ to } n - 1 \text{ do} \)
3. \( A[n - 1 - i] \leftarrow \text{heapDeleteMax}(A) \)

Running time of HeapSort: \( O(n \log n) \)
Selection

**Problem Statement:** The $k$th-max problem asks to find the $k$th largest item in an array $A$ of $n$ numbers.

**Solution 1:** Make $k$ passes through the array, deleting the maximum number each time.

**Complexity:** $\Theta(kn)$.

**Solution 2:** First sort the numbers. Then return the $k$th largest number.

**Complexity:** $\Theta(n \log n)$.

**Solution 3:** Scan the array and maintain the $k$ largest numbers seen so far in a min-heap

**Complexity:** $\Theta(n \log k)$.

**Solution 4:** Make a max-heap by calling *heapify*(A). Call *deleteMax*(A) $k$ times.

**Complexity:** $\Theta(n + k \log n)$. 