Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Abstract Data Types

**Abstract Data Type (ADT):** A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various *realizations* of an ADT, which specify:

- How the information is stored (*data structure*)
- How the operations are performed (*algorithms*)
Priority Queue ADT

**Priority Queue:** An ADT consisting of a collection of items (each having a *priority*) with operations

- **insert:** inserting an item tagged with a priority
- **deleteMax:** removing the item of *highest priority*

*deleteMax* is also called *extractMax*.

Applications: typical “todo” list, simulation systems

The above definition is for a *maximum-oriented* priority queue. A *minimum-oriented* priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin*. 
Realizations of Priority Queues

Attempt 1: Use *unsorted arrays*
Realizations of Priority Queues

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- insert: $O(1)$
- deleteMax: $O(n)$

Using unsorted linked lists is identical.
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Third Realization: Heaps

A *heap* is a certain type of binary tree.

Recall binary trees: A binary tree is either

- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, *etc.*
A *max-heap* is a binary tree with the following two properties:

1. **Structural Property**: All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.

2. **Heap-order Property**: For any node $i$, key (priority) of parent of $i$ is larger than or equal to key of $i$.

A *min-heap* is the same, but with opposite order property.
Heaps

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A *min-heap* is the same, but with opposite order property.

**Lemma:** Height of a heap with $n$ nodes is $\Theta(\log n)$. 
Insertion in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *bubble-up*:
  
  1. while parent(v) exists and key(parent(v)) < key(v) do
  2. swap v and parent(v)
  3. v ← parent(v)

  The new item bubbles up until it reaches its correct place in the heap. Time: $O(height \text{ of heap}) = O(\log n)$. 

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Insertion in Heaps

- Place the new key at the first free leaf
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bubble-up(v)

v: a node of the heap
1. while parent(v) exists and key(parent(v)) < key(v) do
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The new item bubbles up until it reaches its correct place in the heap.
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The new item bubbles up until it reaches its correct place in the heap.

Time: $O(\text{height of heap}) = O(\log n)$. 
deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a *bubble-down*:

\[
\text{bubble-down}(v) \quad \begin{align*}
1. & \text{while } v \text{ is not a leaf} \\
2. & \quad u \leftarrow \text{child of } v \text{ with largest key} \\
3. & \quad \text{if } \text{key}(u) > \text{key}(v) \\
4. & \quad \text{swap } v \text{ and } u \\
5. & \quad v \leftarrow u \\
6. & \text{else} \\
7. & \quad \text{break}
\end{align*}
\]

Time: $O(\text{height of heap}) = O(\log n)$. 

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  v: a node of the heap
  1. while v is not a leaf do
  2.   u ← child of v with largest key
  3.   if key(u) > key(v) then
  4.     swap v and u
  5.   v ← u
  6. else
  7.     break
```

Time: $O(\text{height of heap}) = O(\log n)$.
Priority Queue Realization Using Heaps

**heapInsert**\((A, x)\)

\(A\): an array-based heap, \(x\): a new item

1. \(\text{size}(A) \leftarrow \text{size}(A) + 1\)
2. \(A[\text{size}(A) - 1] \leftarrow x\)
3. \(\text{bubble-up}(A, \text{size}(A) - 1)\)

**heapDeleteMax**\((A)\)

\(A\): an array-based heap

1. \(\text{max} \leftarrow A[0]\)
2. \(\text{swap}(A[0], A[\text{size}(A) - 1])\)
3. \(\text{size}(A) \leftarrow \text{size}(A) - 1\)
4. \(\text{bubble-down}(A, 0)\)
5. \(\text{return} \ max\)

Insert and deleteMax: \(O(\log n)\)
Problem statement: Given $n$ items (in $A[0 \cdots n - 1]$) build a heap containing all of them.
Building Heaps

**Problem statement:** Given $n$ items (in $A[0 \cdots n-1]$) build a heap containing all of them.

**Solution 1:** Start with an empty heap and insert items one at a time:

```
heapify1(A)
A: an array
1. initialize $H$ as an empty heap
2. for $i \leftarrow 0$ to $\text{size}(A) - 1$ do
3. \quad \text{heapInsert}(H, A[i])
```
Building Heaps

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\[
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1. \quad \text{initialize } H \text{ as an empty heap} \\
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3. \quad \textit{heapInsert}(H, A[i])
\]

This corresponds to going from \( 0 \cdots n - 1 \) in \( A \) and doing \textit{bubble-ups}.

Worst-case running time: \( \Theta(n \log n) \).
Storing Heaps in Arrays

Let $H$ be a heap (binary tree) of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements level-by-level from top to bottom, in each level left-to-right.
Storing Heaps in Arrays

Let $H$ be a heap (binary tree) of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

It is easy to find parents and children using this array representation:

- the *left child* of $A[i]$ (if it exists) is $A[2i + 1]$,
- the *right child* of $A[i]$ (if it exists) is $A[2i + 2]$,
- the *parent* of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).
Building Heaps

**Problem statement:** Given $n$ items (in $A[0 \cdots n-1]$) build a heap containing all of them.

**Solution 2:** Using bubble-downs instead:

1. $n \leftarrow \text{size}(A) - 1$
2. for $i \leftarrow \lfloor n / 2 \rfloor$ downto 0 do
3. bubble-down($A$, $i$)

A careful analysis yields a worst-case complexity of $\Theta(n)$.

A heap can be built in linear time.
Building Heaps

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**Solution 2:** Using *bubble-downs* instead:

```plaintext
heapify(A)
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A careful analysis yields a worst-case complexity of $\Theta(n)$. A heap can be built in linear time.
Building Heaps

Problem statement: Given $n$ items (in $A[0 \cdots n - 1]$) build a heap containing all of them.

Solution 2: Using \textit{bubble-downs} instead:

\begin{verbatim}
heapify(A)
A: an array
1. $n \leftarrow \text{size}(A) - 1$
2. \textbf{for} $i \leftarrow \lfloor n/2 \rfloor \textbf{ downto } 0$ \textbf{do}
3. \hspace{0.5cm} \text{bubble-down}(A, i)
\end{verbatim}

A careful analysis yields a worst-case complexity of $\Theta(n)$. A heap can be built in linear time.
Using a Priority Queue to Sort

\[ PQ - Sort(A) \]
1. initialize \( PQ \) to an empty priority queue
2. \( \textbf{for } i \leftarrow 0 \textbf{ to } n - 1 \textbf{ do} \)
3. \( PQ.insert(A[i], A[i]) \)
4. \( \textbf{for } i \leftarrow 0 \textbf{ to } n - 1 \textbf{ do} \)
5. \( A[n - 1 - i] \leftarrow PQ.deleteMax() \)
HeapSort

\(\text{HeapSort}(A)\)
1. initialize \(H\) to an empty heap
2. \textbf{for} \(i \leftarrow 0\) \textbf{to} \(n - 1\) \textbf{do}
3. \(\text{heapInsert}(H, A[i])\)
4. \textbf{for} \(i \leftarrow 0\) \textbf{to} \(n - 1\) \textbf{do}
5. \(A[n - 1 - i] \leftarrow \text{heapDeleteMax}(H)\)

Running time of HeapSort: \(O(n \log n)\)
HeapSort

\textit{HeapSort}(A)
1. initialize \( H \) to an empty heap
2. \textbf{for} \( i \leftarrow 0 \) \textbf{to} \( n - 1 \) \textbf{do}
3. \hspace{1em} \textit{heapInsert}(H, A[i])
4. \textbf{for} \( i \leftarrow 0 \) \textbf{to} \( n - 1 \) \textbf{do}
5. \hspace{1em} \( A[n - 1 - i] \leftarrow \text{heapDeleteMax}(H) \)

\textit{HeapSort}(A)
1. \textit{heapify}(A)
2. \textbf{for} \( i \leftarrow 0 \) \textbf{to} \( n - 1 \) \textbf{do}
3. \hspace{1em} \( A[n - 1 - i] \leftarrow \text{heapDeleteMax}(A) \)

Running time of HeapSort: \( O(n \log n) \)
HeapSort

\textbf{HeapSort}(A)
1. initialize \( H \) to an empty heap
2. \textbf{for} \( i \leftarrow 0 \ \textbf{to} \ n - 1 \ \textbf{do} \\
3. \textbf{heapInsert}(H, A[i]) \\
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\textbf{HeapSort}(A)
1. \textit{heapify}(A) \\
2. \textbf{for} \( i \leftarrow 0 \ \textbf{to} \ n - 1 \ \textbf{do} \\
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Running time of HeapSort: \( O(n \log n) \)
Problem Statement: The $k$th-max problem asks to find the $k$th largest item in an array $A$ of $n$ numbers.

Solution 1: Make $k$ passes through the array, deleting the maximum number each time.
Complexity: $\Theta(kn)$.

Solution 2: First sort the numbers. Then return the $k$th largest number.
Complexity: $\Theta(n \log n)$.

Solution 3: Scan the array and maintain the $k$ largest numbers seen so far in a min-heap
Complexity: $\Theta(n \log k)$.

Solution 4: Make a max-heap by calling $\text{heapify}(A)$. Call $\text{deleteMax}(A)$ $k$ times.
Complexity: $\Theta(n + k \log n)$. 