Module 8: Data Structures for Multi-Dimensional Data

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Multi-Dimensional Data

- Various applications
  - Attributes of a product (laptop: price, screen size, processor speed, RAM, hard drive, · · ·)
  - Attributes of an employee (name, age, salary, · · ·)

- Dictionary for multi-dimensional data
  - A collection of $d$-dimensional items
  - Each item has $d$ aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
  - Operations: insert, delete, range-search query

- (Orthogonal) Range-search query: specify a range (interval) for certain aspects, and find all the items whose aspects fall within given ranges.
  - Example: laptops with screen size between 11 and 13 inches, RAM between 8 and 16 GB, price between 1,500 and 2,000 CAD
Multi-Dimensional Data

- Each item has \( d \) aspects (coordinates): \((x_0, x_1, \cdots, x_{d-1})\)
- Aspect values \((x_i)\) are numbers
- Each item corresponds to a point in \( d \)-dimensional space
- We concentrate on \( d = 2 \), i.e., points in Euclidean plane

![Diagram]

- Item: ordered pair \((x, y) \in \mathbb{R} \times \mathbb{R}\)
- Price (CAD)
- Processor speed (MHz)
- Range-search query \((1350 \leq x \leq 1550, 700 \leq y \leq 1100)\)
- Point: \((1200,1000)\)
One-Dimensional Range Search

- **First solution**: ordered arrays
  - Running time: $O(\log n + k)$, $k$: number of reported items
  - Problem: does not generalize to higher dimensions
- **Second solution**: balanced BST (e.g., AVL tree)

```
BST-RangeSearch(T, k₁, k₂)

T: A balanced search tree, $k₁, k₂$: search keys
Report keys in $T$ that are in range $[k₁, k₂]$

1. if $T = nil$ then return
2. if key($T$) < $k₁$ then
3. BST-RangeSearch($T.right, k₁, k₂$)
4. if key($T$) > $k₂$ then
5. BST-RangeSearch($T.left, k₁, k₂$)
6. if $k₁ ≤ key(T) ≤ k₂$ then
7. BST-RangeSearch($T.left, k₁, k₂$)
8. report key($T$)
9. BST-RangeSearch($T.right, k₁, k₂$)
```
Range Search example

\[ \text{BST-RangeSearch}(T, 30, 65) \]
Range Search example

$\textit{BST-RangeSearch}(T, 30, 65)$

Nodes either on boundary, inside, or outside.
Range Search example

$BST$-RangeSearch($T$, 30, 65)

Nodes either on boundary, inside, or outside.

Note: Not every boundary node is returned.
One-Dimensional Range Search

- $P_1$: path from the root to a leaf that goes right if $k < k_1$ and left otherwise
- $P_2$: path from the root to a leaf that goes left if $k > k_2$ and right otherwise

Partition nodes of $T$ into three groups:

1. **boundary nodes**: nodes in $P_1$ or $P_2$
2. **inside nodes**: non-boundary nodes that belong to either (a subtree rooted at a right child of a node of $P_1$) or (a subtree rooted at a left child of a node of $P_2$)
3. **outside nodes**: non-boundary nodes that belong to either (a subtree rooted at a left child of a node of $P_1$) or (a subtree rooted at a right child of a node of $P_2$)
One-Dimensional Range Search

- **$P_1$**: path from the root to a leaf that goes right if $k < k_1$ and left otherwise
- **$P_2$**: path from the root to a leaf that goes left if $k > k_2$ and right otherwise
- **$k$**: number of reported items
- Nodes visited during the search:
  - $O(\log n)$ boundary nodes
  - $O(k)$ inside nodes
  - No outside nodes
- Running time $O(\log n + k)$
2-Dimensional Range Search

- Each item has 2 aspects (coordinates): \((x_i, y_i)\)
- Each item corresponds to a point in Euclidean plane
- Options for implementing \(d\)-dimensional dictionaries:
  - Reduce to one-dimensional dictionary: combine the \(d\)-dimensional key into one key
    Problem: Range search on one aspect is not straightforward
  - Use several dictionaries: one for each dimension
    Problem: inefficient, wastes space
  - Partition trees
    - A tree with \(n\) leaves, each leaf corresponds to an item
    - Each internal node corresponds to a region
    - quadtrees, kd-trees
  - multi-dimensional range trees
Quadtrees

- We have \( n \) points \( P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\} \) in the plane

- How to build a quadtree on \( P \):
  - Find a square \( R \) that contains all the points of \( P \) (We can compute minimum and maximum \( x \) and \( y \) values among \( n \) points)
  - Root of the quadtree corresponds to \( R \)
  - **Split**: Partition \( R \) into four equal subsquares (quadrants), each correspond to a child of \( R \)
  - Recursively repeat this process for any node that contains more than one point
  - Points on split lines belong to left/bottom side
  - Each leaf stores (at most) one point
  - We can delete a leaf that does not contain any point
Quadtrees

- Example: We have 13 points $P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{12}, y_{12})\}$ in the plane
Quadtrees

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Quadtrees

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Quadtree

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<table>
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</table>
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- Leave

Petrick (SCS, UW)
Quadtrees

- Example: We have 13 points $P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{12}, y_{12})\}$ in the plane
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Quadtree Operations

- **Search**: Analogous to binary search trees
- **Insert**:
  - Search for the point
  - Split the leaf if there are two points
- **Delete**:
  - Search for the point
  - Remove the point
  - If its parent has only one child left, delete that child and continue the process toward the root.
Quadtree: Range Search

\[ QTree-RangeSearch(T, R) \]

\( T \): A quadtree node, \( R \): Query rectangle

1. \textbf{if} \ (T \text{ is a leaf}) \textbf{then}
2. \textbf{if} \ (T\.point \in R) \textbf{then}
3. \quad \text{report} \ T\.point
4. \textbf{for} each child \( C \) of \( T \) \textbf{do}
5. \quad \textbf{if} \ C\.region \cap R \neq \emptyset \textbf{then}
6. \quad \quad QTree-RangeSearch(C, R)

- **spread factor** of points \( P \): \( \beta(P) = \frac{d_{\text{max}}}{d_{\text{min}}} \)
- \( d_{\text{max}}(d_{\text{min}}) \): maximum (minimum) distance between two points in \( P \)
- **height** of quadtree: \( h \in \Theta(\log_2 \frac{d_{\text{max}}}{d_{\text{min}}}) \)
- Complexity to build initial tree: \( \Theta(nh) \)
- Complexity of range search: \( \Theta(nh) \) even if the answer is \( \emptyset \)
Quadtree Conclusion

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (usually the boundary box is padded to get a power of two).
- Space wasteful
- Major drawback: can have very large height for certain nonuniform distributions of points
- Easily generates to higher dimensions (octrees, etc.).
kd-trees

- We have \( n \) points \( P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\} \) in the plane.
- Quadtrees split square into quadrants regardless of where points actually lie.
- kd-tree idea: Split the points into two (roughly) equal subsets.
- How to build a kd-tree on \( P \):
  - Split \( P \) into two equal subsets using a vertical line.
  - Split each of the two subsets into two equal pieces using horizontal lines.
  - Continue splitting, alternating vertical and horizontal lines, until every point is in a separate region.
- Complexity: \( \Theta(n \log n) \), height of the tree: \( \Theta(\log n) \).
kd-trees

- We have $n$ points $P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$ in the plane.
- Quadtrees split square into quadrants regardless of where points actually lie.
- kd-tree idea: Split the points into two (roughly) equal subsets.
- More details:
  - Initially, we sort the $n$ points according to their $x$-coordinates.
  - The root of the tree is the point with median $x$ coordinate (index $\lfloor n/2 \rfloor$ in the sorted list).
  - All other points with $x$ coordinate less than or equal to this go into the left subtree; points with larger $x$-coordinate go in the right subtree.
  - At alternating levels, we sort and split according to $y$-coordinates instead.
- Complexity: $\Theta(n \log n)$, height of the tree: $\Theta(\log n)$
kd-trees

- kd-tree idea: Split the points into two (roughly) equal subsets
- A balanced binary tree

```
   p4
  /   \
 p3   p9
  |    |
 p8   |
     |
p1
      |
 p0
     |
 p2
     |
 p6   p5
     |
 p7
```
kd-trees

- kd-tree idea: Split the points into two (roughly) equal subsets
- A balanced binary tree
kd-trees

- **kd-tree idea:** Split the points into two (roughly) equal subsets
- A **balanced** binary tree

![Diagram of a kd-tree]

- $p_3$, $p_9$
- $p_8$
- $p_0$
- $p_2$
- $p_6$, $p_5$
- $p_7$
- $p_1$
- $p_4$
- $p_8$
- $p_5$
kd-trees

- **kd-tree idea**: Split the points into two (roughly) equal subsets
- A **balanced** binary tree
kd-trees

- kd-tree idea: Split the points into two (roughly) equal subsets
- A balanced binary tree
kd-tree: Range Search

kd-rangeSearch( T, R)

T: A kd-tree node, R: Query rectangle
1. if T is empty then return
2. if T.point ∈ R then
3. report T.point
4. for each child C of T do
5. if C.region ∩ R ≠ ∅ then
6. kd-rangeSearch( C, R)
kd-tree: Range Search

kd-rangeSearch\((T, R, split[← 'x'])\)

\(T\): A kd-tree node, \(R\): Query rectangle

1. \textbf{if} \(T\) is empty \textbf{then} return
2. \textbf{if} \(T\).point \(\in\) \(R\) \textbf{then}
3. \hspace{1em} report \(T\).point
4. \textbf{if} \(split = 'x'\) \textbf{then}
5. \hspace{1em} \textbf{if} \(T\).point.\(x\) \(\geq\) \(R\).leftSide \textbf{then}
6. \hspace{2em} kd-rangeSearch\((T\).left, \(R\), 'y')
7. \hspace{1em} \textbf{if} \(T\).point.\(x\) \(<\) \(R\).rightSide \textbf{then}
8. \hspace{2em} kd-rangeSearch\((T\).right, \(R\), 'y')
9. \textbf{if} \(split = 'y'\) \textbf{then}
10. \hspace{1em} \textbf{if} \(T\).point.\(y\) \(\geq\) \(R\).bottomSide \textbf{then}
11. \hspace{2em} kd-rangeSearch\((T\).left, \(R\), 'x')
12. \hspace{1em} \textbf{if} \(T\).point.\(y\) \(<\) \(R\).topSide \textbf{then}
13. \hspace{2em} kd-rangeSearch\((T\).right, \(R\), 'x')
kd-tree: Range Search Complexity

- The complexity is $O(k + U)$ where $k$ is the number of keys reported and $U$ is the number of regions we go to but unsuccessfully.
- $U$ corresponds to the number of regions which intersect but are not fully in $R$.
- Those regions have to intersect one of the four sides of $R$.
- $Q(n)$: Maximum number of regions in a kd-tree with $n$ points that intersect a vertical (horizontal) line.
- $Q(n)$ satisfies the following recurrence relation:
  \[ Q(n) = 2Q(n/4) + O(1) \]
- It solves to $Q(n) = O(\sqrt{n})$.
- Therefore, the complexity of range search in kd-trees is $O(k + \sqrt{n})$. 
kd-tree: Higher Dimensions

- **kd-trees for** $d$-dimensional space
  - At the root the point set is partitioned based on the first coordinate
  - At the children of the root the partition is based on the second coordinate
  - At depth $d - 1$ the partition is based on the last coordinate
  - At depth $d$ we start all over again, partitioning on first coordinate

- **Storage:** $O(n)$
- **Construction time:** $O(n \log n)$
- **Range query time:** $O(n^{1-1/d} + k)$

(Note: $d$ is considered to be a constant.)
Range Trees

- We have \( n \) points \( P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\} \) in the plane.
- A range tree is a tree of trees (a \textit{multi-level} data structure).
- How to build a range tree on \( P \):
  - Build a balanced binary search tree \( \tau \) determined by the \( x \)-coordinates of the \( n \) points.
  - For every node \( \nu \in \tau \), build a balanced binary search tree \( \tau_{assoc}(\nu) \) (associated structure of \( \tau \)) determined by the \( y \)-coordinates of the nodes in the subtree of \( \tau \) with root node \( \nu \).
Algorithm BUILDRANGE TREE(P)

Input. A set P of points in the plane.

Output. The root of a 2-dimensional range tree.

1. Construct the associated structure: Build a binary search tree Tassoc on the set \( P_y \) of y-coordinates of the points in P. Store at the leaves of Tassoc not just the y-coordinate of the points in \( P_y \), but the points themselves.

2. if \( P \) contains only one point then
   Create a leaf \( \nu \) storing this point, and make Tassoc the associated structure of \( \nu \).

3. else
   Split \( P \) into two subsets; one subset \( P_{\text{left}} \) contains the points with x-coordinate less than or equal to \( x_{\text{mid}} \), the median x-coordinate, and the other subset \( P_{\text{right}} \) contains the points with x-coordinate larger than \( x_{\text{mid}} \).

4. \( \nu_{\text{left}} \leftarrow \text{BUILDRANGE TREE}(P_{\text{left}}) \)

5. \( \nu_{\text{right}} \leftarrow \text{BUILDRANGE TREE}(P_{\text{right}}) \)

6. Create a node \( \nu \) storing \( x_{\text{mid}} \), make \( \nu_{\text{left}} \) the left child of \( \nu \), make \( \nu_{\text{right}} \) the right child of \( \nu \), and make Tassoc the associated structure of \( \nu \).

7. return \( \nu \)

Note that in the leaves of the associated structures we do not just store the y-coordinate of the points but the points themselves. This is important because, when searching the associated structures, we need to report the points and not just the y-coordinates.

Lemma 5.6
A range tree on a set of \( n \) points in the plane requires \( O(n \log n) \) storage.

Proof.
A point \( p \) in \( P \) is stored only in the associated structure of nodes on the path in \( T \) towards the leaf containing \( p \). Hence, for all nodes at a given depth of \( T \),...
Range Trees: Operations

- **Search**: trivially as in a binary search tree
- **Insert**: insert point in $\tau$ by $x$-coordinate
- From inserted leaf, walk back up to the root and insert the point in all associated trees $\tau_{assoc}(v)$ of nodes $v$ on path to the root
- **Delete**: analogous to insertion
- **Note**: re-balancing is a problem!
Range Trees: Range Search

- A two stage process
- To perform a range search query $R = [x_1, x_2] \times [y_1, y_2]$:
  - Perform a range search (on the $x$-coordinates) for the interval $[x_1, x_2]$ in $\tau$ ($\text{BST-RangeSearch}(\tau, x_1, x_2)$)
  - For every outside node, do nothing.
  - For every “top” inside node $v$, perform a range search (on the $y$-coordinates) for the interval $[y_1, y_2]$ in $\tau_{assoc}(v)$. During the range search of $\tau_{assoc}(v)$, do not check any $x$-coordinates (they are all within range).
  - For every boundary node, test to see if the corresponding point is within the region $R$.

- Running time: $O(k + \log^2 n)$
- Range tree space usage: $O(n \log n)$
Range Trees: Higher Dimensions

- Range trees for \( d \)-dimensional space
  - Storage: \( O(n \, (\log n)^{d-1}) \)
  - Construction time: \( O(n \, (\log n)^{d-1}) \)
  - Range query time: \( O((\log n)^d + k) \)

(Note: \( d \) is considered to be a constant.)
Range Trees: Higher Dimensions

- Space/time trade-off
  - **Storage:** \( O(n (\log n)^{d-1}) \)
  - **Construction time:** \( O(n (\log n)^{d-1}) \)
  - **Range query time:** \( O((\log n)^d + k) \)

(\text{Note: } d \text{ is considered to be a constant.})