Module 11: External memory

CS 240 - Data Structures and Data Management

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Different levels of memory

Current architectures:
- registers (very fast, very small)
- cache L1, L2 (still fast, less small)
- main memory
- external memory: disk (slow, very large)

General question: how to adapt our algorithms to take the memory hierarchy into account, avoiding transfers as much as possible?
Review: dictionary ADT

A dictionary is a collection of items, each of which contains
- a key
- some data,
and is called a key-value pair (KVP). Keys can be compared and are
(typically) unique.

Operations:
- search\( (k) \)
- insert\( (k, v) \)
- delete\( (k) \)
- optional: join, isEmpty, size, etc.

Dictionaries in external memory

Tree-based data structures have poor memory locality:
If an operation accesses \( m \) nodes, then it must access
\( m \) spaced-out memory locations.

Observation: Accessing a single location in external memory
(e.g. hard disk) automatically loads a whole block (or “page”).
- In an AVL tree, \( \Theta(\log n) \) pages are loaded in the worst case.
- Better solution: B-trees
2-3 Trees

A 2-3 Tree is like a BST with additional structural properties:
- Every internal node either contains one KVP and two children, or two KVPs and three children.
- The leaves are NIL (do not store keys)
- All the leaves are at the same level.

Searching through a 1-node is just like in a BST.
For a 2-node, we must examine both keys and follow the appropriate path.

Insertion in a 2-3 tree

First, we search to find the lowest internal node where the new key belongs.

If the node has only 1 KVP, just add the new one to make a 2-node.

Otherwise, order the three keys as $a < b < c$.
Split the node into two 1-nodes, containing $a$ and $c$, and (recursively) insert $b$ into the parent along with the new link.
2-3 Tree Insertion

Example:

```
25 43
18
12
NIL NIL
21 24
NIL NIL NIL
31 36
28
NIL NIL
33
NIL NIL
39 42
NIL NIL NIL
51
48
NIL NIL
56 62
NIL NIL NIL
```

Deletion from a 2-3 Tree

As with BSTs and AVL trees, we first swap the KVP with its successor, so that we always delete from a leaf.

Say we’re deleting KVP $x$ from a node $V$:

- If $V$ is a 2-node, just delete $x$.
- Elseif $V$ has a 2-node immediate sibling $U$, perform a transfer:
  Put the “intermediate” KVP in the parent between $V$ and $U$ into $V$, and replace it with the adjacent KVP from $U$.
- Otherwise, we merge $V$ and a 1-node sibling $U$:
  Remove $V$ and (recursively) delete the “intermediate” KVP from the parent, adding it to $U$. 

2-3 Tree Deletion

Example:

```
36
25
18 21
12 19 24
31
28 33
43
41
39 42
51
48 56 62
```

B-Trees

The 2-3 Tree is a specific type of \((a, b)\)-tree:

An \((a, b)\)-tree \emph{of order} \(M\) is a search tree satisfying:

- Each internal node has at least \(a\) children, unless it is the root.
  The root has at least 2 children.
- Each internal node has at most \(b\) children.
- If a node has \(k\) children, then it stores \(k - 1\) key-value pairs (KVPs).
- Leaves store no keys and are at the same level.

A \emph{B-tree of order} \(M\) is a \([M/2], M\)-tree.
A 2-3 tree has \(M = 3\).

\emph{search, insert, delete} work just like for 2-3 trees.
Height of a B-tree

What is the least number of KVPs in a height-\( h \) B-tree?
(Height = # levels not counting the dummy-level −1)

<table>
<thead>
<tr>
<th>Level</th>
<th>Nodes</th>
<th>Links/node</th>
<th>KVP/node</th>
<th>KVPs on level</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>( M/2 )</td>
<td>( M/2 - 1 )</td>
<td>( 2(M/2 - 1) )</td>
</tr>
<tr>
<td>2</td>
<td>( 2(M/2) )</td>
<td>( M/2 )</td>
<td>( M/2 - 1 )</td>
<td>( 2(M/2)(M/2 - 1) )</td>
</tr>
<tr>
<td>3</td>
<td>( 2(M/2)^2 )</td>
<td>( M/2 )</td>
<td>( M/2 - 1 )</td>
<td>( 2(M/2)^2(M/2 - 1) )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>( h )</td>
<td>( 2(M/2)^{h-1} )</td>
<td>( M/2 )</td>
<td>( M/2 - 1 )</td>
<td>( 2(M/2)^{h-1}(M/2 - 1) )</td>
</tr>
</tbody>
</table>

Total: \( n \geq 1 + 2 \sum_{i=0}^{h-1} (M/2)^i(M/2 - 1) = 2(M/2)^h - 1 \)

Therefore height of tree with \( n \) nodes is \( \Theta\left(\left(\log n\right)/\left(\log M\right)\right) \).

Analysis of B-tree operations

Assume each node stores its KVPs and child-pointers in a dictionary that supports \( O(\log M) \) search, insert, and delete.

Then \textit{search}, \textit{insert}, and \textit{delete} work just like for 2-3 trees, and each require \( \Theta(\text{height}) \) node operations.

Total cost is \( O\left(\frac{\log n}{\log M} \cdot (\log M)\right) = O(\log n) \).
Dictionaries in external memory

**Recall**: accessing a single location in *external memory* (e.g. hard disk) automatically loads a whole block (or “page”).

In an AVL tree or 2-3 tree, $\Theta(\log n)$ pages are loaded in the worst case.

If $M$ is small enough so an $M$-node fits into a single page, then a B-tree of order $M$ only loads $\Theta((\log n)/(\log M))$ pages.

This can result in a huge savings: memory access is often the largest time cost in a computation.

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B-tree variations

**Other strategies**: *insert* and *delete* without *backtracking* via *pre-emptive splitting* and *pre-emptive merging*.

**Red-black trees**: Identical to a B-tree with minsize 1 and maxsize 3, but each 2-node or 3-node is represented by 2 or 3 binary nodes, and each node holds a “color” value of red or black.

**$B^+$-trees**: All KVPs are stored at the leaves (interior nodes just have keys), and the leaves are linked sequentially.
Hashing in External Memory

As before, if we have a very large dictionary that must be stored externally, how can we hash and minimize disk transfers?

Say external memory is stored in blocks (or “pages”) of size $S$. Most hash strategies access many pages (data is scattered).

Exception: **Linear Probing**. All hash table accesses will usually be in the same page. But $\alpha$ must be kept small to avoid clustering, so there is a lot of wasted space.

New Idea: **Extendible Hashing**. Similar to a B-tree with height 1 and max size $S$ at the leaves

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**Assumption**: Hash-function has values in $\{0, 1, \ldots, 2^L - 1\}$.

The directory (similar to root node) is stored in *internal memory*. Contains array of size $2^d$, where $d \leq L$ is called the *order*.

Each directory entry points to a *block* stored in *external memory*. Each block contains at most $S$ items. (Many entries can point to the same block.)

To look up a key $k$ in the directory, use the $d$ leading bits of $h(k)$. 

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**Extendible Hashing Overview**

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Extendible Hashing Details

Blocks are shared by entries in a specific manner:
- Every block $B$ stores a local depth $k_B \leq d$.
- Hash values in $B$ agree on leading $k_B$ bits.
- All directory entries with the same $k_B$ leading bits point to $B$.
- So $2^{d-k_B}$ directory entries point to block $B$.

Searching in extendible hashing

Searching is done in the directory, then in a block:
- Given a key $k$, compute $h(k)$.
- Leading $d$ digits of $h(k)$ give index in directory.
- Load block $B$ at this index into main memory.
- Perform a search in $B$ for all items with hash value $h(k)$.
- Search among them for the one with key $k$.

Cost:
- CPU time: depends on how the block are organized (hash table, balanced tree, sorted array)
- Disk transfers: 1 (directory resides in internal memory)
Insertion in Extendible Hashing

\( \text{insert}(k, v) \) is done as follows:

- Search for \( h(k) \) to find the proper block \( B \) for insertion
- If the \( B \) has space, then put \( (k, v) \) there.
- Elseif the block is full and \( k_B < d \), perform a \textit{block split}:
  - Split \( B \) into two blocks \( B_0 \) and \( B_1 \).
  - Separate items according to the \((k_B + 1)\)-th bit.
  - Set local depth in \( B_0 \) and \( B_1 \) to \( k_B + 1 \)
  - Update references in the directory
  - Try again to insert
- Elseif the block is full and \( k_B = d \), perform a \textit{directory grow}:
  - Double the size of the directory \( (d \leftarrow d + 1) \)
  - Update references appropriately.
  - Then split block \( B \) (which is now possible).

Extendible hashing insert example with \( S = 2 \)

\begin{align*}
\text{Insert}(00100) & \quad \text{Insert}(01010)
\end{align*}
Extendible hashing conclusion

*delete*(k) is performed in a reverse manner to *insert*:
- Search for block B and remove k from it
- If block becomes too empty, then we perform a *block merge*
- If every block B has local depth \( k_B \leq d - 1 \), perform a *directory shrink*

But most likely just do *lazy deletion*.

Cost of *insert* and *delete*:
- CPU time: Search in a block depends on the implementation \( \Theta(S) \) to do/undo one split
  - Directory grow/shrink costs \( \Theta(2^d) \) (but very rare).
- Disk transfers: 1 when no split

Summary of extendible hashing

- Directory is much smaller than total number of stored keys and should fit in main memory.
- To make more space, we only add one block.
  - Rarely do we have to change the size of the directory.
  - Never do we have to move all items in the dictionary (in contrast to normal hashing).
- Space usage is not too inefficient: can be shown that under uniform hashing, each block is expected to be 69% full.
- Potentially extra CPU cost