Module 11: External memory

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Different levels of memory

Current architectures:
- registers (very fast, very small)
- cache L1, L2 (still fast, less small)
- main memory
- external memory: disk (slow, very large)

General question: how to adapt our algorithms to take the memory hierarchy into account, avoiding transfers as much as possible?
Review: dictionary ADT

A *dictionary* is a collection of *items*, each of which contains

- a *key*
- some *data*,

and is called a *key-value pair* (KVP). Keys can be compared and are (typically) unique.

Operations:

- \textit{search}(k)
- \textit{insert}(k, v)
- \textit{delete}(k)
- optional: \textit{join}, \textit{isEmpty}, \textit{size}, etc.
Dictionaries in external memory

Tree-based data structures have poor *memory locality*: If an operation accesses \( m \) nodes, then it must access \( m \) spaced-out memory locations.

**Observation**: Accessing a single location in *external memory* (e.g. hard disk) automatically loads a whole block (or “page”).

- In an AVL tree, \( \Theta(\log n) \) pages are loaded in the worst case.
- Better solution: B-trees
2-3 Trees

A 2-3 Tree is like a BST with additional structural properties:
- Every internal node either contains one KVP and two children, or two KVPs and three children.
- The leaves are NIL (do not store keys).
- All the leaves are at the same level.

Searching through a 1-node is just like in a BST. For a 2-node, we must examine both keys and follow the appropriate path.
Insertion in a 2-3 tree

First, we search to find the lowest internal node where the new key belongs.

If the node has only 1 KVP, just add the new one to make a 2-node.

Otherwise, order the three keys as $a < b < c$. Split the node into two 1-nodes, containing $a$ and $c$, and (recursively) insert $b$ into the parent along with the new link.
2-3 Tree Insertion

Example: \textit{insert}(19)
2-3 Tree Insertion

**Example:** \textit{insert}(19)
2-3 Tree Insertion

Example: \textit{insert}(19)

```
25 43

18

12 19 21 24

28

31 36

33

39 42

48

51

56 62
```
2-3 Tree Insertion

Example: *insert*(19)
2-3 Tree Insertion

Example: $insert(41)$

(NIL-leaves not shown to simplify picture)
2-3 Tree Insertion

Example: \textit{insert}(41)

(NIL-leaves not shown to simplify picture)
2-3 Tree Insertion

**Example:** \textit{insert}(41)

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2-3 Tree Insertion

Example: \textit{insert}(41)

(NIL-leaves not shown to simplify picture)
2-3 Tree Insertion

Example: insert(41)

(NIL-leaves not shown to simplify picture)
Deletion from a 2-3 Tree

As with BSTs and AVL trees, we first swap the KVP with its successor, so that we always delete from a leaf.

Say we’re deleting KVP $x$ from a node $V$:

- If $V$ is a 2-node, just delete $x$.
- Elseif $V$ has a 2-node immediate sibling $U$, perform a transfer: Put the “intermediate” KVP in the parent between $V$ and $U$ into $V$, and replace it with the adjacent KVP from $U$.
- Otherwise, we merge $V$ and a 1-node sibling $U$: Remove $V$ and (recursively) delete the “intermediate” KVP from the parent, adding it to $U$. 
2-3 Tree Deletion

Example: \textit{delete}(43)
2-3 Tree Deletion

Example: \textit{delete}(43)
2-3 Tree Deletion

**Example:** \( \text{delete}(43) \)
2-3 Tree Deletion

Example: $delete(19)$
2-3 Tree Deletion

Example: \( \text{delete}(19) \)
2-3 Tree Deletion

**Example:** \textit{delete}(19)
2-3 Tree Deletion

Example: \textit{delete}(42)
2-3 Tree Deletion

Example: \textit{delete}(42)
Example: \textit{delete}(42)
2-3 Tree Deletion

**Example:** `delete(42)`
2-3 Tree Deletion

**Example:** $\text{delete}(42)$

```
  25  36
 /    /
18    31
|     |
12    28
```

```
  48  56
 /    /
39    41
|    /|
33   51
|  /  |
21  62
```
B-Trees

The 2-3 Tree is a specific type of \((a, b)\)-tree:

An \((a, b)\)-tree of order \(M\) is a search tree satisfying:

- Each internal node has at least \(a\) children, unless it is the root. The root has at least 2 children.
- Each internal node has at most \(b\) children.
- If a node has \(k\) children, then it stores \(k - 1\) key-value pairs (KVPs).
- Leaves store no keys and are at the same level.

A B-tree of order \(M\) is a \(\left\lceil M/2 \right\rceil, M\)-tree.
A 2-3 tree has \(M = 3\).

search, insert, delete work just like for 2-3 trees.
# Height of a B-tree

What is the least number of KVPs in a height-$h$ B-tree? (Height = \# levels not counting the dummy-level – 1)

<table>
<thead>
<tr>
<th>Level</th>
<th>Nodes</th>
<th>Links/node</th>
<th>KVP/node</th>
<th>KVPs on level</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>$M/2$</td>
<td>$M/2 - 1$</td>
<td>$2(M/2 - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$2(M/2)$</td>
<td>$M/2$</td>
<td>$M/2 - 1$</td>
<td>$2(M/2)(M/2 - 1)$</td>
</tr>
<tr>
<td>3</td>
<td>$2(M/2)^2$</td>
<td>$M/2$</td>
<td>$M/2 - 1$</td>
<td>$2(M/2)^2(M/2 - 1)$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>h</td>
<td>$2(M/2)^{h-1}$</td>
<td>$M/2$</td>
<td>$M/2 - 1$</td>
<td>$2(M/2)^{h-1}(M/2 - 1)$</td>
</tr>
</tbody>
</table>

Total: $n \geq 1 + 2 \sum_{i=0}^{h-1} (M/2)^i(M/2 - 1) = 2(M/2)^h - 1$

Therefore height of tree with $n$ nodes is $\Theta((\log n)/(\log M))$. 
Analysis of B-tree operations

Assume each node stores its KVPs and child-pointers in a dictionary that supports $O(\log M)$ search, insert, and delete.

Then search, insert, and delete work just like for 2-3 trees, and each require $\Theta(height)$ node operations.

Total cost is $O \left( \frac{\log n}{\log M} \cdot (\log M) \right) = O(\log n)$. 
Recall: accessing a single location in *external memory* (e.g. hard disk) automatically loads a whole block (or “page”).

In an AVL tree or 2-3 tree, $\Theta(\log n)$ pages are loaded in the worst case.

If $M$ is small enough so an $M$-node fits into a single page, then a B-tree of order $M$ only loads $\Theta((\log n)/(\log M))$ pages.

This can result in a huge savings: memory access is often the largest time cost in a computation.
B-tree variations

Other strategies: insert and delete without backtracking via pre-emptive splitting and pre-emptive merging.

Red-black trees: Identical to a B-tree with minsize 1 and maxsize 3, but each 2-node or 3-node is represented by 2 or 3 binary nodes, and each node holds a “color” value of red or black.

B⁺-trees: All KVPs are stored at the leaves (interior nodes just have keys), and the leaves are linked sequentially.
Hashing in External Memory

As before, if we have a very large dictionary that must be stored externally, how can we hash and minimize disk transfers?

Say external memory is stored in blocks (or “pages”) of size $S$. Most hash strategies access many pages (data is scattered).

Exception: **Linear Probing**. All hash table accesses will usually be in the same page. But $\alpha$ must be kept small to avoid clustering, so there is a lot of wasted space.

New Idea: **Extendible Hashing**. Similar to a B-tree with height 1 and max size $S$ at the leaves.
Assumption: Hash-function has values in \( \{0, 1, \ldots, 2^L - 1\} \).

The directory (similar to root node) is stored in internal memory.
Contains array of size \( 2^d \), where \( d \leq L \) is called the order.

Each directory entry points to a block stored in external memory.
Each block contains at most \( S \) items. (Many entries can point to the same block.)

To look up a key \( k \) in the directory, use the \( d \) leading bits of \( h(k) \).
Blocks are shared by entries in a specific manner:

- Every block $B$ stores a local depth $k_B \leq d$.
- Hash values in $B$ agree on leading $k_B$ bits.
- All directory entries with the same $k_B$ leading bits point to $B$.
- So $2^{d-k_B}$ directory entries point to block $B$. 
Searching in extendible hashing

Searching is done in the directory, then in a block:

- Given a key $k$, compute $h(k)$.
- Leading $d$ digits of $h(k)$ give index in directory.
- Load block $B$ at this index into main memory.
- Perform a search in $B$ for all items with hash value $h(k)$.
- Search among them for the one with key $k$. 

Cost:
- CPU time: depends on how the blocks are organized (hash table, balanced tree, sorted array)
- Disk transfers: 1 (directory resides in internal memory)
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\textit{insert}(k, v) is done as follows:

- Search for \( h(k) \) to find the proper block \( B \) for insertion
- If the \( B \) has space, then put \( (k, v) \) there.
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- If the \( B \) has space, then put \((k, v)\) there.
- Elself the block is full and \( k_B < d \), perform a \textit{block split}:
  - Split \( B \) into two blocks \( B_0 \) and \( B_1 \).
  - Separate items according to the \((k_B + 1)\)-th bit.
  - Set local depth in \( B_0 \) and \( B_1 \) to \( k_B + 1 \).
  - Update references in the directory.
  - Try again to insert.
- Elself the block is full and \( k_B = d \), perform a \textit{directory grow}:
  - Double the size of the directory (\( d \leftarrow d + 1 \)).
  - Update references appropriately.
  - Then split block \( B \) (which is now possible).
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Extendible hashing insert example with $S = 2$

- **Insert( 00100 )**
- **Insert( 01010 )**
Extendible hashing insert example with $S = 2$

$d=2$

- **Insert (00100)**
- **Insert (01010)**

$d=3$

- **Insert (00001)**
- **Insert (01000)**
Extendible hashing conclusion

\textit{delete}(k) is performed in a reverse manner to \textit{insert}:

- Search for block \( B \) and remove \( k \) from it
- If block becomes too empty, then we perform a \textit{block merge}
- If every block \( B \) has local depth \( k_B \leq d - 1 \), perform a \textit{directory shrink}

But most likely just do \textit{lazy deletion}. 

\textbf{Cost of insert and delete:}

- \textbf{CPU time:} Search in a block depends on the implementation \( \Theta(S) \) to do/undo one split
- \textbf{Directory grow/shrink costs} \( \Theta(2^d) \) (but very rare).
- \textbf{Disk transfers:} 1 when no split
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Summary of extendible hashing

- Directory is much smaller than total number of stored keys and should fit in main memory.
- To make more space, we only add one block. Rarely do we have to change the size of the directory. *Never* do we have to move all items in the dictionary (in contrast to normal hashing).
- Space usage is not too inefficient: can be shown that under uniform hashing, each block is expected to be 69% full.
- Potentially extra CPU cost