Module 2: Priority Queues

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Based on lecture notes by many previous cs240 instructors

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References: Sedgewick 9.1-9.4
Outline

1. Priority Queues
   - Abstract Data Types
   - ADT Priority Queue
   - Binary Heaps
   - Operations in Binary Heaps
   - \textit{PQ-sort} and \textit{Heapsort}
   - Towards the Selection Problem
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Abstract Data Types

Abstract Data Type (ADT): A description of information and a collection of operations on that information.

The information is accessed only through the operations.

We can have various realizations of an ADT, which specify:
- How the information is stored (data structure)
- How the operations are performed (algorithms)
Stack ADT

**Stack**: an ADT consisting of a collection of items with operations:
- *push*: inserting an item
- *pop*: removing (and typically returning) the most recently inserted item

Items are removed in LIFO (last-in first-out) order.
Items enter the stack at the *top* and are removed from the *top*.
We can have extra operations: *size*, *isEmpty*, and *top*

Applications: Addresses of recently visited web sites, procedure calls

Realizations of Stack ADT
- using arrays
- using linked lists
Queue ADT

**Queue**: an ADT consisting of a collection of items with operations:
- *enqueue*: inserting an item
- *dequeue*: removing (and typically returning) the least recently inserted item

Items are removed in FIFO (*first-in first-out*) order. Items enter the queue at the *rear* and are removed from the *front*. We can have extra operations: *size*, *isEmpty*, and *front*

Applications: Waiting lines, printer queues

Realizations of Queue ADT
- using (circular) arrays
- using linked lists
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Priority Queue ADT

**Priority Queue:** An ADT consisting of a collection of items (each having a **priority**) with operations

- **insert:** inserting an item tagged with a priority
- **deleteMax:** removing and returning the item of **highest** priority

*deleteMax* is also called *extractMax* or *getmax*. The priority is also called *key*.

The above definition is for a **maximum-oriented** priority queue. A **minimum-oriented** priority queue is defined in the natural way, replacing operation *deleteMax* by *deleteMin*,

Applications: typical “todo” list, simulation systems, sorting
## Using a Priority Queue to Sort

### PQ-Sort $A[0..n-1]$)

1. initialize $PQ$ to an empty priority queue
2. for $i \leftarrow 0$ to $n - 1$ do
   3. $PQ.insert(A[i])$
3. for $i \leftarrow n - 1$ down to 0 do
   4. $A[i] \leftarrow PQ.deleteMax()$

- Note: Run-time depends on how we implement the priority queue.
- Sometimes written as: $O(initialization + n \cdot insert + n \cdot deleteMax)$
Realizations of Priority Queues

**Realization 1**: unsorted arrays

- **insert**: $O(1)$
- **deleteMax**: $O(n)$

*Note: We assume dynamic arrays, i.e., expand by doubling as needed. (Amortized over all insertions this takes $O(1)$ extra time.)*

Using unsorted linked lists is identical.

PQ-sort with this realization yields selection sort.

**Realization 2**: sorted arrays

- **insert**: $O(n)$
- **deleteMax**: $O(1)$

Using sorted linked lists is identical.

PQ-sort with this realization yields insertion sort.
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**Realization 2**: sorted arrays

- *insert*: $O(n)$
- *deleteMax*: $O(1)$

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Realization 3: Heaps

A **(binary) heap** is a certain type of binary tree.

You should know:

- A **binary tree** is either
  - empty, or
  - consists of three parts: a node and two binary trees (left subtree and right subtree).
- **Terminology:** root, leaf, parent, child, level, sibling, ancestor, descendant, etc.
- Any binary tree with $n$ nodes has height at least $\log(n + 1) - 1 \in \Omega(\log n)$. 
Example Heap

In our examples we only show the priorities, and we show them directly in the node. A more accurate picture would be:

```
( priority = 50, <other info> )
```
A **heap** is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.

2. **Heap-order Property:** For any node $i$, the key of the parent of $i$ is larger than or equal to key of $i$. 

The full name for this is *max-oriented binary heap*. 

**Lemma:** The height of a heap with $n$ nodes is $\Theta(\log n)$. 

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Heaps – Definition

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The full name for this is *max-oriented binary heap*.

**Lemma**: The height of a heap with $n$ nodes is $\Theta(\log n)$. 
Storing Heaps in Arrays

Heaps should *not* be stored as binary trees!

Let $H$ be a heap of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.
Storing Heaps in Arrays

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Heaps in Arrays – Navigation

It is easy to navigate the heap using this array representation:

- the root node is at index 0 (We use “node” and “index” interchangeably in this implementation.)
- the left child of node $i$ (if it exists) is node $2i + 1$
- the right child of node $i$ (if it exists) is node $2i + 2$
- the parent of node $i$ (if it exists) is node $\lfloor \frac{i-1}{2} \rfloor$
- the last node is $n – 1$
Heaps in Arrays – Navigation

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  (We use “node” and “index” interchangeably in this implementation.)
- the **left child** of node \( i \) (if it exists) is node \( 2i + 1 \)
- the **right child** of node \( i \) (if it exists) is node \( 2i + 2 \)
- the **parent** of node \( i \) (if it exists) is node \( \left\lfloor \frac{i-1}{2} \right\rfloor \)
- the **last** node is \( n - 1 \)

We should hide implementation details using helper-functions!

- functions **root()**, **parent(i)**, **last(n)**, etc.
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**Insert in Heaps**

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *fix-up*:
Insert in Heaps

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\[
\text{fix-up}(A, i) \\
i: \text{an index corresponding to a node of the heap} \\
1. \textbf{while} parent(i) \text{ exists and } A[parent(i)].key < A[i].key \textbf{ do} \\
2. \quad \text{swap } A[i] \text{ and } A[parent(i)] \\
3. \quad i \leftarrow parent(i)
\]

The new item “bubbles up” until it reaches its correct place in the heap.
### Insert in Heaps

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\]

The new item “bubbles up” until it reaches its correct place in the heap.

**Time:** $O(\text{height of heap}) = O(\log n)$. 
fix-up example

```
      50
     / \  /  \\
    29  34  27
     /  /  /  \\
    23 26 15  8
```

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fix-up example
fix-up example
fix-up example
deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a \textit{fix-down}:

```plaintext
1. while i is not a leaf do
2. // Find the child with the larger key
3. j ← left child of i
4. if (j is not last (n) and A[j+1].key > A[j].key) then
5. j ← j + 1
6. if A[i].key ≥ A[j].key then break
7. swap A[j] and A[i]
8. i ← j
```

Time: $O(\text{height of heap}) = O(\log n)$.
deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a fix-down:

```plaintext
fix-down(A, n, i)
A: an array that stores a heap of size n
i: an index corresponding to a node of the heap
1. while i is not a leaf do
2.    // Find the child with the larger key
3.    j ← left child of i
4.    if (j is not last(n) and A[j + 1].key > A[j].key)
5.       j ← j + 1
6.    if A[i].key ≥ A[j].key break
7.    swap A[j] and A[i]
8.    i ← j
```

Time: $O(\text{height of heap}) = O(\log n)$.
deleteMax example
deleteMax example
deleteMax example
deleteMax example

```
48
  / \
29   34
  /   / \
27   15   8  10
  /   /   /   /
23   26   8    10
```

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Priority Queue Realization Using Heaps

Store items in array $A$ and globally keep track of $size$

\[
\text{insert}(x) \\
1. \text{ increase } size \\
2. \ell \leftarrow \text{last}(size) \\
3. A[\ell] \leftarrow x \\
4. \text{ fix-up}(A, \ell)
\]

\[
deleteMax() \\
1. \ell \leftarrow \text{last}(size) \\
2. \text{ swap } A[\text{root()}] \text{ and } A[\ell] \\
3. \text{ decrease } size \\
4. \text{ fix-down}(A, size, \text{root()}) \\
5. \text{ return } A[\ell]
\]

\textit{insert} and \textit{deleteMax}: $O(\log n)$
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Sorting using heaps

- Recall: Any priority queue can be used to sort in time

\[ O(\text{initialization} + n \cdot \text{insert} + n \cdot \text{deleteMax}) \]

- Using the binary-heaps implementation of PQs, we obtain:

\[
\text{PQsortWithHeaps}(A)
\]
\begin{itemize}
  \item initialize \( H \) to an empty heap
  \item for \( i \leftarrow 0 \) to \( n - 1 \) do
  \item \( H.\text{insert}(A[i]) \)
  \item for \( i \leftarrow n - 1 \) down to 0 do
  \item \( A[i] \leftarrow H.\text{deleteMax}() \)
\end{itemize}
Sorting using heaps

- Recall: Any priority queue can be used to sort in time

\[ O(\text{initialization} + n \cdot \text{insert} + n \cdot \text{deleteMax}) \]

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  \]

- both operations run in \( O(\log n) \) time for heaps

\( \leadsto \) PQ-Sort using heaps takes \( O(n \log n) \) time.

- Can improve this with two simple tricks \( \rightarrow \) Heapsort

  1. Heaps can be built faster if we know all input in advance.
  2. Can use the same array for input and heap. \( \leadsto O(1) \) auxiliary space!
Building Heaps with Fix-up

**Problem:** Given $n$ items all at once (in $A[0 \cdots n - 1]$) build a heap containing all of them.
Building Heaps with Fix-up

**Problem:** Given $n$ items all at once (in $A[0 \cdots n - 1]$) build a heap containing all of them.

**Solution 1:** Start with an empty heap and insert items one at a time:

```
simpleHeapBuilding(A)
A: an array
1. initialize $H$ as an empty heap
2. for $i \leftarrow 0$ to \textbf{size}(A) – 1 do
3. \hspace{1em} $H.insert(A[i])$
```

*Worst-case running time: $\Theta(n \log n)$.\*
Building Heaps with Fix-up

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```

This corresponds to doing *fix-ups*

Worst-case running time: $\Theta(n \log n)$. 
Building Heaps with Fix-down

**Problem:** Given $n$ items all at once (in $A[0 \cdots n - 1]$) build a heap containing all of them.
Building Heaps with Fix-down

**Problem:** Given \( n \) items all at once (in \( A[0 \cdots n - 1] \)) build a heap containing all of them.

**Solution 2:** Using *fix-downs* instead:

```plaintext
heapify(A)
A: an array
1. \( n \leftarrow A.size() \)
2. \( \text{for } i \leftarrow \text{parent(last}(n)) \) \text{ downto } 0 \text{ do}
3. \( \text{fix-down}(A, n, i) \)
```

A careful analysis yields a worst-case complexity of \( \Theta(n) \).

A heap can be built in linear time.
Building Heaps with Fix-down

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heapify example
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HeapSort

- Idea: *PQ-sort* with heaps.
- $O(1)$ auxiliary space: Use same input-array $A$ for storing heap.

```
HeapSort(A, n)
1. // heapify
2. n ← A.size()
3. for i ← parent(last(n)) downto 0 do
4.   fix-down(A, n, i)
5. // repeatedly find maximum
6. while n > 1
7.   // delete the maximum
8.   swap items at A[root()] and A[last(n)]
9.   decrease n
10. fix-down(A, n, root())
```

The for-loop takes $\Theta(n)$ time and the while-loop takes $O(n \log n)$ time.
Heapsort example

Continue with the example from heapify:
Heapsort example

Continue with the example from heapify:

```
30  70  40  10  80  20  60  50  10
```

The array (i.e., the heap in level-by-level order) is now in sorted order.
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10
10
30
70

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50
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```

The array (i.e., the heap in level-by-level order) is now in sorted order.
Heap summary

- **Binary heap**: A binary tree that satisfies structural property and heap-order property.
- Heaps are one possible realization of ADT PriorityQueue:
  - `insert` takes time $O(\log n)$
  - `deleteMax` takes time $O(\log n)$
  - Also supports `findMax` in time $O(1)$
- A binary heap can be built in linear time.
- **PQ-sort** with binary heaps leads to a sorting algorithm with $O(n \log n)$ worst-case run-time (⇝ **HeapSort**)
- We have seen here the max-oriented version of heaps (the maximum priority is at the root).
- There exists a symmetric min-oriented version that supports `insert` and `deleteMin` with the same run-times.
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Finding the largest items

**Problem:** Find the *kth largest item* in an array $A$ of $n$ distinct numbers.

**Solution 1:** Make $k$ passes through the array, deleting the maximum number each time.
Complexity: $\Theta(kn)$.

**Solution 2:** Sort $A$, then return $A[n-k]$.
Complexity: $\Theta(n \log n)$.

**Solution 3:** Scan the array and maintain the $k$ largest numbers seen so far in a min-heap
Complexity: $\Theta(n \log k)$.

**Solution 4:** Create a max-heap with $heapify(A)$. Call $deleteMax(A)$ $k$ times.
Complexity: $\Theta(n + k \log n)$. 