Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Abstract Data Types

**Abstract Data Type (ADT):** A description of information and a collection of operations on that information.

The information is accessed only through the operations.

We can have various realizations of an ADT, which specify:

- How the information is stored (data structure)
- How the operations are performed (algorithms)

Priority Queue ADT

**Priority Queue:** An ADT consisting of a collection of items (each having a priority) with operations

- *insert:* inserting an item tagged with a priority
- *deleteMax:* removing the item of highest priority

*deleteMax* is also called *extractMax.*

Applications: typical “todo” list, simulation systems

The above definition is for a maximum-oriented priority queue. A minimum-oriented priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin.*
Realizations of Priority Queues

Attempt 1: Use unsorted arrays
- insert: $O(1)$
- deleteMax: $O(n)$

Using unsorted linked lists is identical.

Attempt 2: Use sorted arrays
- insert: $O(n)$
- deleteMax: $O(1)$

Using sorted linked-lists is identical.

Third Realization: Heaps

A heap is a certain type of binary tree.

Recall binary trees:
A binary tree is either
- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.

Heaps

A max-heap is a binary tree with the following two properties:

- Structural Property: All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are left-justified.
- Heap-order Property: For any node $i$, key (priority) of parent of $i$ is larger than or equal to key of $i$.

A min-heap is the same, but with opposite order property.

Lemma: Height of a heap with $n$ nodes is $\Theta (\log n)$. 
Insertion in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a bubble-up:

```plaintext
bubble-up(v)
v: a node of the heap
1. while parent(v) exists and key(parent(v)) < key(v) do
2. swap v and parent(v)
3. v ← parent(v)
```

The new item bubbles up until it reaches its correct place in the heap.

Time: \(O(\text{height of heap}) = O(\log n)\).

DeleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a bubble-down:

```plaintext
bubble-down(v)
v: a node of the heap
1. while v is not a leaf do
2. u ← child of v with largest key
3. if key(u) > key(v) then
4. swap v and u
5. v ← u
6. else
7. break
```

Time: \(O(\text{height of heap}) = O(\log n)\).

Priority Queue Realization Using Heaps

`heapInsert(A, x)`

- \(A\): an array-based heap, \(x\): a new item
- \(1.\) size\((A)\) ← size\((A) + 1\)
- \(2.\) \(A[size(A) - 1]\) ← \(x\)
- \(3.\) bubble-up\((A, size(A) - 1)\)

`heapDeleteMax(A)`

- \(A\): an array-based heap
- \(1.\) max ← \(A[0]\)
- \(2.\) swap\((A[0], A[size(A) - 1])\)
- \(3.\) size\((A)\) ← size\((A) - 1\)
- \(4.\) bubble-down\((A, 0)\)
- \(5.\) return max

Insert and deleteMax: \(O(\log n)\).
Storing Heaps in Arrays

Let $H$ be a heap (binary tree) of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements level-by-level from top to bottom, in each level left-to-right.

It is easy to find parents and children using this array representation:
- the left child of $A[i]$ (if it exists) is $A[2i+1]$,
- the right child of $A[i]$ (if it exists) is $A[2i+2]$,
- the parent of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).

Building Heaps

Problem statement: Given $n$ items (in $A[0 \cdot \cdot n - 1]$) build a heap containing all of them.

Solution 1: Start with an empty heap and insert items one at a time:

```
heapify1(A)
A: an array
1. initialize $H$ as an empty heap
2. for $i \leftarrow 0$ to $\text{size}(A) - 1$ do
3. heapInsert($H, A[i]$)
```

This corresponds to going from $0 \cdot \cdot n - 1$ in $A$ and doing bubble-ups
Worst-case running time: $\Theta(n \log n)$.

Solution 2: Using bubble-downs instead:

```
heapify(A)
A: an array
1. $n \leftarrow \text{size}(A) - 1$
2. for $i \leftarrow \lfloor n/2 \rfloor$ downto 0 do
3. bubble-down($A, i$)
```

A careful analysis yields a worst-case complexity of $\Theta(n)$.
A heap can be built in linear time.
Using a Priority Queue to Sort

**PQ − Sort(A)**
1. initialize PQ to an empty priority queue
2. for i ← 0 to n − 1 do
3.   PQ.insert(A[i], A[i])
4. for i ← 0 to n − 1 do
5.   A[n − 1 − i] ← PQ.deleteMax()

HeapSort

**HeapSort(A)**
1. initialize H to an empty heap
2. for i ← 0 to n − 1 do
3.   heapInsert(H, A[i])
4. for i ← 0 to n − 1 do
5.   A[n − 1 − i] ← heapDeleteMax(H)

**HeapSort(A)**
1. heapify(A)
2. for i ← 0 to n − 1 do
3.   A[n − 1 − i] ← heapDeleteMax(A)

Running time of HeapSort: \(O(n \log n)\)

Selection

**Problem Statement:** The kth-max problem asks to find the kth largest item in an array A of n numbers.

**Solution 1:** Make k passes through the array, deleting the maximum number each time.

**Complexity:** \(\Theta(kn)\).

**Solution 2:** First sort the numbers. Then return the kth largest number.

**Complexity:** \(\Theta(n \log n)\).

**Solution 3:** Scan the array and maintain the k largest numbers seen so far in a min-heap

**Complexity:** \(\Theta(n \log k)\).

**Solution 4:** Make a max-heap by calling heapify(A). Call deleteMax(A) k times.

**Complexity:** \(\Theta(n + k \log n)\).