Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Abstract Data Types

Abstract Data Type (ADT): A description of information and a collection of operations on that information.

The information is accessed only through the operations.

We can have various realizations of an ADT, which specify:

- How the information is stored (data structure)
- How the operations are performed (algorithms)

Priority Queue ADT

Priority Queue: An ADT consisting of a collection of items (each having a priority) with operations

- insert: inserting an item tagged with a priority
- deleteMax: removing the item of highest priority

deleteMax is also called extractMax.

Applications: typical “todo” list, simulation systems

The above definition is for a maximum-oriented priority queue. A minimum-oriented priority queue is defined in the natural way, by replacing the operation deleteMax by deleteMin.

Realizations of Priority Queues

Attempt 1: Use unsorted arrays

- insert: $O(1)$
- deleteMax: $O(n)$

Using unsorted linked lists is identical.

Attempt 2: Use sorted arrays

- insert: $O(n)$
- deleteMax: $O(1)$

Using sorted linked-lists is identical.
Third Realization: Heaps

A heap is a certain type of binary tree.

Recall binary trees:
A binary tree is either
- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.

Heaps

A max-heap is a binary tree with the following two properties:
1. Structural Property: All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are left-justified.
2. Heap-order Property: For any node $i$, key (priority) of parent of $i$ is larger than or equal to key of $i$.

A min-heap is the same, but with opposite order property.

Lemma: Height of a heap with $n$ nodes is $\Theta(\log n)$.

Insertion in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a bubble-up:

  bubble-up($v$)
  $v$: a node of the heap
  1. while parent($v$) exists and key(parent($v$)) < key($v$) do
  2. swap $v$ and parent($v$)
  3. $v \leftarrow$ parent($v$)

The new item bubbles up until it reaches its correct place in the heap.

Time: $O(\text{height of heap}) = O(\log n)$.

deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a bubble-down:

  bubble-down($v$)
  $v$: a node of the heap
  1. while $v$ is not a leaf do
  2. $u \leftarrow$ child of $v$ with largest key
  3. if key($u$) > key($v$) then
  4. swap $v$ and $u$
  5. $v \leftarrow u$
  6. else
  7. break

Time: $O(\text{height of heap}) = O(\log n)$. 
**Priority Queue Realization Using Heaps**

```latex
heapInsert(A, x)
A: an array-based heap, x: a new item
1. size(A) ← size(A) + 1
2. A[size(A) − 1] ← x
3. bubble-up(A, size(A) − 1)
```

**heapDeleteMax(A)**

A: an array-based heap
1. max ← A[0]
2. swap(A[0], A[size(A) − 1])
3. size(A) ← size(A) − 1
4. bubble-down(A, 0)
5. return max

Insert and deleteMax: $O(\log n)$

**Storing Heaps in Arrays**

Let $H$ be a heap (binary tree) of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements level-by-level from top to bottom, in each level left-to-right.

It is easy to find parents and children using this array representation:
- the left child of $A[i]$ (if it exists) is $A[2i + 1]$,
- the right child of $A[i]$ (if it exists) is $A[2i + 2]$,
- the parent of $A[i]$ ($i \neq 0$) is $A[\lfloor i - 1 \rfloor 2]$ ($A[0]$ is the root node).

**Building Heaps**

**Problem statement:** Given $n$ items (in $A[0 \ldots n − 1]$) build a heap containing all of them.

**Solution 1:** Start with an empty heap and insert items one at a time:

```latex
heapify1(A)
A: an array
1. initialize $H$ as an empty heap
2. for $i \leftarrow 0$ to $size(A) − 1$ do
3. heapInsert($H$, $A[i]$)
```

This corresponds to going from $0 \ldots n − 1$ in $A$ and doing bubble-ups
Worst-case running time: $\Theta(n \log n)$.

**Solution 2:** Using bubble-downs instead:

```latex
heapify(A)
A: an array
1. $n \leftarrow size(A) − 1$
2. for $i \leftarrow \lfloor n/2 \rfloor$ downto 0 do
3. bubble-down($A$, $i$)
```

A careful analysis yields a worst-case complexity of $\Theta(n)$. A heap can be built in linear time.
Using a Priority Queue to Sort

\[ PQ - \text{Sort}(A) \]
1. initialize \( PQ \) to an empty priority queue
2. for \( i \leftarrow 0 \) to \( n - 1 \) do
   3. \( PQ.\text{insert}(A[i], A[i]) \)
   4. for \( i \leftarrow 0 \) to \( n - 1 \) do
      5. \( A[n - 1 - i] \leftarrow PQ.\text{deleteMax}() \)

HeapSort

\[ \text{HeapSort}(A) \]
1. initialize \( H \) to an empty heap
2. for \( i \leftarrow 0 \) to \( n - 1 \) do
   3. \( \text{heapInsert}(H, A[i]) \)
   4. for \( i \leftarrow 0 \) to \( n - 1 \) do
      5. \( A[n - 1 - i] \leftarrow \text{heapDeleteMax}(H) \)

Running time of HeapSort: \( O(n \log n) \)

Selection

**Problem Statement:** The \( k \)th-max problem asks to find the \( k \)th largest item in an array \( A \) of \( n \) numbers.

**Solution 1:** Make \( k \) passes through the array, deleting the maximum number each time.
*Complexity:* \( \Theta(kn) \).

**Solution 2:** First sort the numbers. Then return the \( k \)th largest number.
*Complexity:* \( \Theta(n \log n) \).

**Solution 3:** Scan the array and maintain the \( k \) largest numbers seen so far in a min-heap
*Complexity:* \( \Theta(n \log k) \).

**Solution 4:** Make a max-heap by calling \( \text{heapify}(A) \). Call \( \text{deleteMax}(A) \) \( k \) times.
*Complexity:* \( \Theta(n + k \log n) \).