Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Abstract Data Types

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various *realizations* of an ADT, which specify:

- How the information is stored (*data structure*)
- How the operations are performed (*algorithms*)
Priority Queue ADT

**Priority Queue:** An ADT consisting of a collection of items (each having a priority) with operations

- *insert:* inserting an item tagged with a priority
- *deleteMax:* removing the item of highest priority

*deleteMax* is also called *extractMax*.

Applications: typical “todo” list, simulation systems

The above definition is for a *maximum-oriented* priority queue. A *minimum-oriented* priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin*. 
Realizations of Priority Queues

Attempt 1: Use *unsorted arrays*

- insert: $O(1)$
- deleteMax: $O(n)$

Using unsorted linked lists is identical.

Attempt 2: Use *sorted arrays*

- insert: $O(n)$
- deleteMax: $O(1)$

Using sorted linked-lists is identical.
Third Realization: Heaps

A *heap* is a certain type of binary tree.

Recall binary trees:
A binary tree is either

- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc. .
A **max-heap** is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.

2. **Heap-order Property:** For any node $i$, key (priority) of parent of $i$ is larger than or equal to key of $i$.

A **min-heap** is the same, but with opposite order property.

**Lemma:** Height of a heap with $n$ nodes is $\Theta (\log n)$.
Insertion in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a bubble-up:

\[
\text{bubble-up}(v)
\]
\[
v: \text{a node of the heap}
\]
1. \textbf{while} parent\((v)\) exists \textbf{and} \text{key(parent\((v)\))} < \text{key\((v)\)} \textbf{do}
2. \text{swap } v \text{ and parent\((v)\)}
3. \[v \leftarrow \text{parent\((v)\)}\]

The new item bubbles up until it reaches its correct place in the heap.

Time: \(O(\text{height of heap}) = O(\log n)\).
deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a bubble-down:

```
bubble-down(v)
v: a node of the heap
1. while v is not a leaf do
2. u ← child of v with largest key
3. if key(u) > key(v) then
4. swap v and u
5. v ← u
6. else
7. break
```

Time: $O(\text{height of heap}) = O(\log n)$. 
Priority Queue Realization Using Heaps

\[
\text{heapInsert}(A, x)
\]

A: an array-based heap, \( x \): a new item
1. \( \text{size}(A) \leftarrow \text{size}(A) + 1 \)
2. \( A[\text{size}(A) - 1] \leftarrow x \)
3. \( \text{bubble-up}(A, \text{size}(A) - 1) \)

\[
\text{heapDeleteMax}(A)
\]

A: an array-based heap
1. \( \text{max} \leftarrow A[0] \)
2. \( \text{swap}(A[0], A[\text{size}(A) - 1]) \)
3. \( \text{size}(A) \leftarrow \text{size}(A) - 1 \)
4. \( \text{bubble-down}(A, 0) \)
5. \( \text{return} \ \text{max} \)

Insert and deleteMax: \( O(\log n) \)
Storing Heaps in Arrays

Let $H$ be a heap (binary tree) of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements level-by-level from top to bottom, in each level left-to-right.

It is easy to find parents and children using this array representation:

- the left child of $A[i]$ (if it exists) is $A[2i + 1]$,
- the right child of $A[i]$ (if it exists) is $A[2i + 2]$,
- the parent of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).
Building Heaps

**Problem statement:** Given $n$ items (in $A[0 \cdot \cdot n - 1]$) build a heap containing all of them.

**Solution 1:** Start with an empty heap and insert items one at a time:

```
heapify1(A)
A: an array
1. initialize $H$ as an empty heap
2. for $i \leftarrow 0$ to size$(A) - 1$ do
3. heapInsert($H$, $A[i]$)
```

This corresponds to going from $0 \cdot \cdot n - 1$ in $A$ and doing *bubble-ups*
Worst-case running time: $\Theta(n \log n)$.
Building Heaps

**Problem statement:** Given $n$ items (in $A[0 \cdots n-1]$) build a heap containing all of them.

**Solution 2:** Using *bubble-downs* instead:

```plaintext
heapify(A)
A: an array
1.   $n \leftarrow \text{size}(A) - 1$
2.   for $i \leftarrow \lfloor n/2 \rfloor$ downto 0 do
3.      bubble-down(A, i)
```

A careful analysis yields a worst-case complexity of $\Theta(n)$. A heap can be built in linear time.
Using a Priority Queue to Sort

\[ PQ - Sort(A) \]
1. initialize \( PQ \) to an empty priority queue
2. \( \textbf{for } i \leftarrow 0 \textbf{ to } n - 1 \textbf{ do} \)
3. \( PQ\text{.insert}(A[i], A[i]) \)
4. \( \textbf{for } i \leftarrow 0 \textbf{ to } n - 1 \textbf{ do} \)
5. \( A[n - 1 - i] \leftarrow PQ\text{.deleteMax}() \)
HeapSort

\[
\text{HeapSort}(A)
\]
1. initialize \( H \) to an empty heap
2. \textbf{for} \( i \leftarrow 0 \) \textbf{to} \( n - 1 \) \textbf{do}
3. \( \text{heapInsert}(H, A[i]) \)
4. \textbf{for} \( i \leftarrow 0 \) \textbf{to} \( n - 1 \) \textbf{do}
5. \( A[n - 1 - i] \leftarrow \text{heapDeleteMax}(H) \)

Running time of HeapSort: \( O(n \log n) \)
Selection

Problem Statement: The $k$th-max problem asks to find the $k$th largest item in an array $A$ of $n$ numbers.

Solution 1: Make $k$ passes through the array, deleting the maximum number each time.
Complexity: $\Theta(kn)$.

Solution 2: First sort the numbers. Then return the $k$th largest number.
Complexity: $\Theta(n \log n)$.

Solution 3: Scan the array and maintain the $k$ largest numbers seen so far in a min-heap
Complexity: $\Theta(n \log k)$.

Solution 4: Make a max-heap by calling $heapify(A)$. Call $deleteMax(A)$ $k$ times.
Complexity: $\Theta(n + k \log n)$.