Module 2: Priority Queues

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Abstract Data Types

**Abstract Data Type (ADT):** A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various *realizations* of an ADT, which specify:
- How the information is stored (*data structure*)
- How the operations are performed (*algorithms*)
Priority Queue ADT

**Priority Queue:** An ADT consisting of a collection of items (each having a *priority*) with operations

- *insert:* inserting an item tagged with a priority
- *deleteMax:* removing the item of *highest priority*

DELETEMAX is also called *extractMax.*

Applications: typical “todo” list, simulation systems

The above definition is for a *maximum-oriented* priority queue. A *minimum-oriented* priority queue is defined in the natural way, by replacing the operation *deleteMax* by *deleteMin.*
Realizations of Priority Queues

Attempt 1: Use *unsorted arrays*

Attempt 2: Use *sorted arrays*
Realizations of Priority Queues

Attempt 1: Use *unsorted arrays*

- insert: $O(1)$
- deleteMax: $O(n)$

Using unsorted linked lists is identical.
Realizations of Priority Queues

Attempt 1: Use *unsorted arrays*

- **insert**: $O(1)$
- **deleteMax**: $O(n)$

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Attempt 2: Use *sorted arrays*

- **insert**: $O(n)$
- **deleteMax**: $O(1)$
Realizations of Priority Queues

Attempt 1: Use *unsorted arrays*

- insert: $O(1)$
- deleteMax: $O(n)$

Using unsorted linked lists is identical.

Attempt 2: Use *sorted arrays*

- insert: $O(n)$
- deleteMax: $O(1)$

Using sorted linked-lists is identical.
A heap is a certain type of binary tree.

Recall binary trees:
A binary tree is either
- empty, or
- consists of three parts: a node and two binary trees (left subtree and right subtree).

Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.
Heaps

A \textit{max-heap} is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are \textit{left-justified}.

2. **Heap-order Property:** For any node \(i\), key (priority) of parent of \(i\) is larger than or equal to key of \(i\).

A \textit{min-heap} is the same, but with opposite order property.
Heaps

A **max-heap** is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.

2. **Heap-order Property:** For any node $i$, key (priority) of parent of $i$ is larger than or equal to key of $i$.

A **min-heap** is the same, but with opposite order property.

**Lemma:** Height of a heap with $n$ nodes is $\Theta (\log n)$. 
Insertion in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *bubble-up*:

\[
\text{bubble-up}(v) \\
\begin{align*}
&1. \text{while } \text{parent}(v) \text{ exists and } \text{key}(\text{parent}(v)) < \text{key}(v) \\
&2. \text{swap } v \text{ and } \text{parent}(v) \\
&3. v \leftarrow \text{parent}(v)
\end{align*}
\]

The new item bubbles up until it reaches its correct place in the heap.

Time: $O(\text{height of heap}) = O(\log n)$. 
Insertion in Heaps

- Place the new key at the first free leaf
- The heap-order property might be violated: perform a *bubble-up*:

   \[
   \text{bubble-up}(v) \\
   v: \text{a node of the heap} \\
   1. \textbf{while } \text{parent}(v) \text{ exists and } \text{key}(\text{parent}(v)) < \text{key}(v) \textbf{ do} \\\n   2. \text{swap } v \text{ and } \text{parent}(v) \\\n   3. v \leftarrow \text{parent}(v)
   \]

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Insertion in Heaps

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\begin{quote}
\begin{tabular}{l}
\textbf{bubble-up}(v) \\
v: a node of the heap \\
1. \textbf{while} parent(v) exists \textbf{and} key(parent(v)) < key(v) \textbf{do} \\
2. swap v and parent(v) \\
3. v \leftarrow parent(v)
\end{tabular}
\end{quote}

The new item bubbles up until it reaches its correct place in the heap.

Time: $O(\text{height of heap}) = O(\log n)$. 
deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a *bubble-down*:

```plaintext
while v is not a leaf do
    u ← child of v with largest key
    if key(u) > key(v) then
        swap v and u
        v ← u
    else
        break
Time: O(height of heap) = O(log n).
```
deleteMax in Heaps

- The maximum item of a heap is just the root node.
- We replace root by the last leaf (last leaf is taken out).
- The heap-order property might be violated: perform a bubble-down:

\[
\text{bubble-down}(v) \\
v: \text{a node of the heap} \\
1. \text{while } v \text{ is not a leaf do} \\
2. \quad u \leftarrow \text{child of } v \text{ with largest key} \\
3. \quad \text{if } \text{key}(u) > \text{key}(v) \text{ then} \\
4. \quad \text{swap } v \text{ and } u \\
5. \quad v \leftarrow u \\
6. \quad \text{else} \\
7. \quad \text{break}
\]

Time: $O(\text{height of heap}) = O(\log n)$. 

Priority Queue Realization Using Heaps

**heapInsert** \((A, x)\)

\(A\): an array-based heap, \(x\): a new item

1. \(\text{size}(A) \leftarrow \text{size}(A) + 1\)
2. \(A[\text{size}(A) - 1] \leftarrow x\)
3. \(\text{bubble-up}(A, \text{size}(A) - 1)\)

**heapDeleteMax** \((A)\)

\(A\): an array-based heap

1. \(\text{max} \leftarrow A[0]\)
2. \(\text{swap}(A[0], A[\text{size}(A) - 1])\)
3. \(\text{size}(A) \leftarrow \text{size}(A) - 1\)
4. \(\text{bubble-down}(A, 0)\)
5. \(\text{return} \max\)

Insert and deleteMax: \(O(\log n)\)
Storing Heaps in Arrays

Let $H$ be a heap (binary tree) of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.
Storing Heaps in Arrays

Let $H$ be a heap (binary tree) of $n$ items and let $A$ be an array of size $n$. Store root in $A[0]$ and continue with elements level-by-level from top to bottom, in each level left-to-right.

It is easy to find parents and children using this array representation:

- the *left child* of $A[i]$ (if it exists) is $A[2i + 1]$,
- the *right child* of $A[i]$ (if it exists) is $A[2i + 2]$,
- the *parent* of $A[i]$ ($i \neq 0$) is $A[\lfloor \frac{i-1}{2} \rfloor]$ ($A[0]$ is the root node).
Problem statement: Given $n$ items (in $A[0 \cdots n - 1]$) build a heap containing all of them.
Building Heaps

**Problem statement:** Given \( n \) items (in \( A[0 \cdots n - 1] \)) build a heap containing all of them.

**Solution 1:** Start with an empty heap and insert items one at a time:

```plaintext
heapify1(A)
A: an array
1. initialize \( H \) as an empty heap
2. for \( i \leftarrow 0 \) to \( \text{size}(A) - 1 \) do
3. \( \text{heapInsert}(H, A[i]) \)
```

Worst-case running time: \( \Theta(n \log n) \).
Building Heaps

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**Solution 1:** Start with an empty heap and insert items one at a time:

```
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A: an array
1. initialize $H$ as an empty heap
2. for $i \leftarrow 0$ to $size(A) - 1$ do
3.   heapInsert($H, A[i]$)
```

This corresponds to going from $0 \cdots n - 1$ in $A$ and doing **bubble-ups**

Worst-case running time: $\Theta(n \log n)$.  

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Building Heaps

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Building Heaps

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Solution 2: Using \textit{bubble-downs} instead:

\begin{verbatim}
heapify(A)
A: an array
1. \( n \leftarrow \text{size}(A) - 1 \)
2. for \( i \leftarrow \lfloor n/2 \rfloor \) downto 0 do
3. \hspace{1em} \text{bubble-down}(A, i)
\end{verbatim}
Building Heaps

**Problem statement:** Given \( n \) items (in \( A[0 \cdots n-1] \)) build a heap containing all of them.

**Solution 2:** Using *bubble-downs* instead:

```plaintext
heapify(A)
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1. \( n \leftarrow \text{size}(A) - 1 \)
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```

A careful analysis yields a worst-case complexity of \( \Theta(n) \).
A heap can be built in linear time.
Using a Priority Queue to Sort

\[
PQ - \text{Sort}(A)
\]
1. initialize \( PQ \) to an empty priority queue
2. \( \text{for } i \leftarrow 0 \text{ to } n - 1 \text{ do} \)
3. \( PQ.\text{insert}(A[i], A[i]) \)
4. \( \text{for } i \leftarrow 0 \text{ to } n - 1 \text{ do} \)
5. \( A[n - 1 - i] \leftarrow PQ.\text{deleteMax}() \)
HeapSort

**HeapSort**($A$)

1. initialize $H$ to an empty heap
2. for $i ← 0$ to $n − 1$ do
3. $heapInsert(H, A[i])$
4. for $i ← 0$ to $n − 1$ do
5. $A[n − 1 − i] ← heapDeleteMax(H)$
HeapSort

HeapSort(A)
1. initialize H to an empty heap
2. for i ← 0 to n − 1 do
3. heapInsert(H, A[i])
4. for i ← 0 to n − 1 do
5. A[n − 1 − i] ← heapDeleteMax(H)

HeapSort(A)
1. heapify(A)
2. for i ← 0 to n − 1 do
3. A[n − 1 − i] ← heapDeleteMax(A)
HeapSort

**HeapSort**(A)
1. initialize $H$ to an empty heap
2. for $i \leftarrow 0$ to $n - 1$ do
3. \hspace{1em} heapInsert($H, A[i]$)
4. for $i \leftarrow 0$ to $n - 1$ do
5. \hspace{1em} $A[n - 1 - i] \leftarrow$ heapDeleteMax($H$)

**HeapSort**(A)
1. heapify($A$)
2. for $i \leftarrow 0$ to $n - 1$ do
3. \hspace{1em} $A[n - 1 - i] \leftarrow$ heapDeleteMax($A$)

Running time of HeapSort: $O(n \log n)$
Selection

**Problem Statement:** The $k$th-max problem asks to find the $k$th largest item in an array $A$ of $n$ numbers.

**Solution 1:** Make $k$ passes through the array, deleting the maximum number each time.

**Complexity:** $\Theta(kn)$.

**Solution 2:** First sort the numbers. Then return the $k$th largest number.

**Complexity:** $\Theta(n \log n)$.

**Solution 3:** Scan the array and maintain the $k$ largest numbers seen so far in a min-heap.

**Complexity:** $\Theta(n \log k)$.

**Solution 4:** Make a max-heap by calling $\text{heapify}(A)$. Call $\text{deleteMax}(A)$ $k$ times.

**Complexity:** $\Theta(n + k \log n)$. 