Module 4: Dictionaries and Balanced Search Trees

CS 240 - Data Structures and Data Management

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Dictionary ADT

A dictionary is a collection of items, each of which contains
- a key
- some data,
and is called a key-value pair (KVP). Keys can be compared and are (typically) unique.

Operations:
- search(k)
- insert(k, v)
- delete(k)
- optional: join, isEmpty, size, etc.
Elementary Implementations

Common assumptions:

- Dictionary has $n$ KVPs
- Each KVP uses constant space
- Comparing keys takes constant time

**Unordered array or linked list**

- search $\Theta(n)$
- insert $\Theta(1)$
- delete $\Theta(n)$ (need to search)

**Ordered array**

- search $\Theta(\log n)$
- insert $\Theta(n)$
- delete $\Theta(n)$

Binary Search Trees (review)

Structure A BST is either empty or contains a KVP, left child BST, and right child BST.

Ordering Every key $k$ in $T.left$ is less than the root key.

Every key $k$ in $T.right$ is greater than the root key.
BST Search and Insert

\[ \text{search}(k) \] Compare \( k \) to current node, stop if found, else recurse on subtree unless it’s empty

\[ \text{insert}(k, v) \] Search for \( k \), then insert \((k, v)\) as new node

Example:

BST Delete

- If node is a leaf, just delete it.
- If node has one child, move child up
- Else, swap with successor or predecessor node and then delete
Height of a BST

*search, insert, delete* all have cost $\Theta(h)$, where

$h = \text{height of the tree} = \text{max. path length from root to leaf}$

If $n$ items are *inserted* one-at-a-time, how big is $h$?

- **Worst-case:**
- **Best-case:**
- **Average-case:**

AVL Trees

Introduced by Adel’son-Vel’skiĭ and Landis in 1962, an *AVL Tree* is a BST with an additional structural property:

The heights of the left and right subtree differ by at most 1.

(The height of an empty tree is defined to be $-1$.)

At each non-empty node, we store $\text{height}(R) - \text{height}(L) \in \{-1, 0, 1\}$:

- $-1$ means the tree is *left-heavy*
- $0$ means the tree is *balanced*
- $1$ means the tree is *right-heavy*

- We could store the actual height, but storing balances is simpler and more convenient.
AVL insertion

To perform $\text{insert}(T, k, v)$:
- First, insert $(k, v)$ into $T$ using usual BST insertion
- Then, move up the tree from the new leaf, updating balance factors.
- If the balance factor is $-1$, $0$, or $1$, then keep going.
- If the balance factor is $\pm 2$, then call the $\text{fix}$ algorithm
to “rebalance” at that node. We are done.

How to “fix” an unbalanced AVL tree

Goal: change the structure without changing the order

Notice that if heights of $A, B, C, D$ differ by at most 1,
then the tree is a proper AVL tree.
Right Rotation

This is a right rotation on node z:

```
  z
 / 
/   
A  y
|   |
|   |
B   x

A  B
C  D
```

**Note**: Only two edges need to be moved, and two balances updated. Useful to fix left-left imbalance.

Again . . .
Left Rotation

This is a \textit{left rotation} on node $z$:

Again, only two edges need to be moved and two balances updated. Useful to fix right-right imbalance.

Again . . .
Pseudocode for rotations

**rotate-right(T)**

- **T**: AVL tree
- returns rotated AVL tree
  1. `newroot ← T.left`
  2. `T.left ← newroot.right`
  3. `newroot.right ← T`
  4. `return newroot`

**rotate-left(T)**

- **T**: AVL tree
- returns rotated AVL tree
  1. `newroot ← T.right`
  2. `T.right ← newroot.left`
  3. `newroot.left ← T`
  4. `return newroot`

Double Right Rotation

This is a *double right rotation* on node z:

First, a left rotation on the left subtree (y). Second, a right rotation on the whole tree (z).
Useful for left-right imbalance.
Double Left Rotation

This is a *double left rotation* on node z:

![Diagram showing double left rotation]

Right rotation on right subtree (y), followed by left rotation on the whole tree (z).
Useful for right-left imbalance.
Fixing a slightly-unbalanced AVL tree

**Idea:** Identify one of the previous 4 situations, apply rotations

\[
\text{fix}(T) \\
T: \text{AVL tree with } T.balance = \pm 2 \\
\text{returns a balanced AVL tree} \\
1. \quad \text{if } T.balance = -2 \text{ then} \\
2. \quad \quad \quad \text{if } T.left.balance = 1 \text{ then} \\
3. \quad \quad \quad \quad T.left \leftarrow \text{rotate-left}(T.left) \\
4. \quad \quad \quad \text{return } \text{rotate-right}(T) \\
5. \quad \text{else if } T.balance = 2 \text{ then} \\
6. \quad \quad \quad \text{if } T.right.balance = -1 \text{ then} \\
7. \quad \quad \quad \quad T.right \leftarrow \text{rotate-right}(T.right) \\
8. \quad \quad \quad \text{return } \text{rotate-left}(T)
\]

AVL Tree Operations

**search:** Just like in BSTs, costs $\Theta(height)$

**insert:** Shown already, total cost $\Theta(height)$

- fix restores the height of the tree it fixes to what it was,
- so fix will be called at most once.

**delete:** First search, then swap with successor (as with BSTs), then move up the tree and apply fix (as with insert).

- fix may be called $\Theta(height)$ times.

Total cost is $\Theta(height)$. 
AVL tree examples

Example:

Height of an AVL tree

Define $N(h)$ to be the least number of nodes in a height-$h$ AVL tree.

One subtree must have height at least $h - 1$, the other at least $h - 2$:

$$N(h) = \begin{cases} 
1 + N(h - 1) + N(h - 2), & h \geq 1 \\
1, & h = 0 \\
0, & h = -1 
\end{cases}$$

What sequence does this look like?
AVL Tree Analysis

Easier lower bound on $N(h)$:

$$N(h) > 2N(h-2) > 4N(h-4) > 8N(h-6) > \cdots > 2^i N(h-2i) \geq 2^\left\lceil h/2 \right\rceil$$

Since $n > 2^\left\lceil h/2 \right\rceil$, $h \leq 2 \log n$,
and thus an AVL tree with $n$ nodes has height $O(\log n)$.
Also, $n \leq 2^{h+1} - 1$, so the height is $\Theta(\log n)$.

$\Rightarrow$ search, insert, delete all cost $\Theta(\log n)$. 