Module 5: Dictionaries II

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Dictionary ADT: Review

A dictionary is a collection of key-value pairs (KVPs), supporting operations search, insert, and delete.

Realizations

- **Unordered array or linked list**: $\Theta(1)$ insert, $\Theta(n)$ search and delete
- **Ordered array**: $\Theta(\log n)$ search, $\Theta(n)$ insert and delete
- **Balanced search trees** (AVL trees):
  - $\Theta(\log n)$ search, insert, and delete
Self-Organizing Search

- Unordered linked list
  - search: $\Theta(n)$, insert: $\Theta(1)$, delete: $\Theta(1)$ (after a search)
- Linear search to find an item in the list
- Is there a more useful ordering?
  - No: if items are accessed equally likely
  - Yes: otherwise (we have a probability distribution for items)

**Optimal static ordering:** sorting items by their probabilities of access in non-increasing order minimizes the expected cost of Search.

**Proof Idea:** For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.

Optimal Static Ordering

A list of elements ordered by non-increasing probability of access has minimum expected access cost

- $L = \langle x_1, x_2, \ldots, x_n \rangle$
  - Expected access cost in $L$ is
  - $E(L) = \sum_{i=1}^{n} P(x_i)T(x_i) = \sum_{i=1}^{n} P(x_i) \cdot i$
  - $P(x_i)$ - access probability for $x_i$
  - $T(x_i)$ - position of $x_i$ in $L$

**Example**
- $P(a) = 0.3$ $P(b) = 0.5$ $P(c) = 0.2$
- $L = \langle a, b, c \rangle$
- $E(L) = 0.3 + 0.5 \cdot 2 + 0.2 \cdot 3 = 1.9$
- $L = \langle b, a, c \rangle$
- $E(L) = 0.5 + 0.3 \cdot 2 + 0.2 \cdot 3 = 1.7$
Optimal Static Ordering

A list of elements ordered by non-increasing probability of access has minimum expected access cost

Proof by Contradiction

$L = \langle x_1, \ldots, x_k, x_{k+1}, \ldots, x_n \rangle$

Suppose the access cost of $L$ is optimal and there is $k$ such that $P(x_k) < P(x_{k+1})$

\[
E(L) = P(x_k) \cdot k + P(x_{k+1}) \cdot (k + 1) + \sum_{i \neq k, k+1} P(x_i) \cdot i
\]

Create another list $L'$ by swapping $x_k$ and $x_{k+1}$.

$L' = \langle x_1, \ldots, x_{k+1}, x_k, \ldots, x_n \rangle$

\[
E(L') = P(x_{k+1}) \cdot k + P(x_k) \cdot (k + 1) + \sum_{i \neq k, k+1} P(x_i) \cdot i
\]

$E(L') - E(L) = P(x_k) - P(x_{k+1}) < 0 \Rightarrow E(L') < E(L)$

Contradiction

Dynamic Ordering

What if we do not know the access probabilities ahead of time?

**Move-To-Front (MTF):** Upon a successful search, move the accessed item to the front of the list

**Transpose:** Upon a successful search, swap the accessed item with the item immediately preceding it

Performance of dynamic ordering:

- Both can be implemented in arrays or linked lists.
- Transpose does not adapt quickly to changing access patterns.
- MTF Works well in practice.
- Theoretically MTF is “competitive”:
  No more than twice as bad as the optimal “offline” ordering.
Skip Lists

- **Randomized** data structure for dictionary ADT
- A hierarchy of ordered linked lists
- A **skip list** for a set $S$ of items is a series of lists $S_0, S_1, \ldots, S_h$ such that:
  - Each list $S_i$ contains the special keys $-\infty$ and $+\infty$
  - List $S_0$ contains the keys of $S$ in non-decreasing order
  - Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
  - List $S_h$ contains only the two special keys

![Skip List Diagram]

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Skip Lists

- A **skip list** for a set $S$ of items is a series of lists $S_0, S_1, \ldots, S_h$
- A two-dimensional collection of positions: **levels** and **towers**
- Traversing the skip list: after($p$), below($p$)
Search in Skip Lists

```plaintext
skip-search(L, k)
L: A skip list, k: a key
1. \( p \leftarrow \) topmost left position of L
2. \( S \leftarrow \) stack of positions, initially containing \( p \)
3. \textbf{while} \( \text{below}(p) \neq \text{null} \) \textbf{do}
4. \( p \leftarrow \text{below}(p) \)
5. \textbf{while} \( \text{key}(\text{after}(p)) < k \) \textbf{do}
6. \( p \leftarrow \text{after}(p) \)
7. push \( p \) onto \( S \)
8. \textbf{return} \( S \)
```

- \( S \) contains positions of the largest key \textbf{less than} \( k \) at each level.
- \( \text{after}(\text{top}(S)) \) will have key \( k \), iff \( k \) is in \( L \).
- \textbf{drop down}: \( p \leftarrow \text{below}(p) \)
- \textbf{scan forward}: \( p \leftarrow \text{after}(p) \)

Example: Skip-Search(\( S \), 87)
Insert in Skip Lists

**Skip-Insert**($S, k, v$)

- Randomly compute the height of new item: repeatedly toss a coin until you get tails, let $i$ be the number of times the coin came up heads
- Search for $k$ in the skip list and find the positions $p_0, p_1, \cdots, p_i$ of the items with largest key less than $k$ in each list $S_0, S_1, \cdots, S_i$ (by performing **Skip-Search**($S, k$))
- Insert item ($k, v$) into list $S_j$ after position $p_j$ for $0 \leq j \leq i$ (a tower of height $i$)

Example: **Skip-Insert**($S, 52, v$)

![Diagram of Skip-Lists](image)
Insert in Skip Lists

Example: Skip-Insert(\(S, 100, v\))

\[ S_3 \quad -\infty \quad +\infty \]
\[ S_2 \quad -\infty \quad 65 \quad +\infty \]
\[ S_1 \quad -\infty \quad 37 \quad 65 \quad 83 \quad +\infty \]
\[ S_0 \quad -\infty \quad 23 \quad 37 \quad 44 \quad 65 \quad 69 \quad 79 \quad 83 \quad 87 \quad 94 \quad +\infty \]

Delete in Skip Lists

Skip-Delete(\(S, k\))

- Search for \(k\) in the skip list and find all the positions \(p_0, p_1, \ldots, p_i\) of the items with the largest key smaller than \(k\), where \(p_j\) is in list \(S_j\). (this is the same as Skip-Search)
- For each \(i\), if \(\text{key}(\text{after}(p_i)) = k\), then remove \text{after}(p_i) from list \(S_i\)
- Remove all but one of the lists \(S_i\) that contain only the two special keys
Delete in Skip Lists

Example: Skip-Delete(S, 65)

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Summary of Skip Lists

- **Expected space usage**: $O(n)$
- **Expected height**: $O(\log n)$
- A skip list with $n$ items has height at most $3 \log n$ with probability at least $1 - 1/n^2$
- **Skip-Search**: $O(\log n)$ expected time
- **Skip-Insert**: $O(\log n)$ expected time
- **Skip-Delete**: $O(\log n)$ expected time
- Skip lists are fast and simple to implement in practice