Module 5: Dictionaries II

CS 240 - Data Structures and Data Management

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Dictionary ADT: Review

A dictionary is a collection of key-value pairs (KVPs), supporting operations search, insert, and delete.

Realizations

- Unordered array or linked list: $\Theta(1)$ insert, $\Theta(n)$ search and delete
- Ordered array: $\Theta(\log n)$ search, $\Theta(n)$ insert and delete
- Balanced search trees (AVL trees):
  - $\Theta(\log n)$ search, insert, and delete

Self-Organizing Search

- Unordered linked list
  - search: $\Theta(n)$, insert: $\Theta(1)$, delete: $\Theta(1)$ (after a search)
- Linear search to find an item in the list
- Is there a more useful ordering?
  - No: if items are accessed equally likely
  - Yes: otherwise (we have a probability distribution for items)
- Optimal static ordering: sorting items by their probabilities of access in non-increasing order minimizes the expected cost of Search.
- Proof Idea: For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.
Optimal Static Ordering

A list of elements ordered by non-increasing probability of access has minimum expected access cost

- \( L = \langle x_1, x_2, \ldots, x_n \rangle \)
- Expected access cost in \( L \) is
  \[ E(L) = \sum_{i=1}^{n} P(x_i) T(x_i) = \sum_{i=1}^{n} P(x_i) \cdot i \]
  \( P(x_i) \) - access probability for \( x_i \)
  \( T(x_i) \) - position of \( x_i \) in \( L \)

- Example
  \( P(a) = 0.3 \)
  \( P(b) = 0.5 \)
  \( P(c) = 0.2 \)
  \( L = \langle a, b, c \rangle \)
  \[ E(L) = 0.3 + 0.5 \cdot 2 + 0.2 \cdot 3 = 1.9 \]

Optimal Static Ordering

A list of elements ordered by non-increasing probability of access has minimum expected access cost

Proof by Contradiction

- \( L = \langle x_1, \ldots, x_k, x_{k+1}, \ldots, x_n \rangle \)
  Suppose the access cost of \( L \) is optimal and there is \( k \) such that
  \( P(x_k) < P(x_{k+1}) \)
  \[ E(L) = P(x_k) \cdot k + P(x_{k+1}) \cdot (k + 1) + \sum_{i \neq k, k+1} P(x_i) \cdot i \]

- Create another list \( L' \) by swapping \( x_k \) and \( x_{k+1} \).
  \( L' = \langle x_1, \ldots, x_{k+1}, x_k, \ldots, x_n \rangle \)
  \[ E(L') = P(x_{k+1}) \cdot k + P(x_k) \cdot (k + 1) + \sum_{i \neq k, k+1} P(x_i) \cdot i \]

- \( E(L') - E(L) = P(x_k) - P(x_{k+1}) < 0 \Rightarrow E(L') < E(L) \)
  Contradiction

Dynamic Ordering

- What if we do not know the access probabilities ahead of time?
- Move-To-Front (MTF): Upon a successful search, move the accessed item to the front of the list
- Transpose: Upon a successful search, swap the accessed item with the item immediately preceding it

Performance of dynamic ordering:

- Both can be implemented in arrays or linked lists.
- Transpose does not adapt quickly to changing access patterns.
- MTF Works well in practice.
- Theoretically MTF is “competitive”:
  No more than twice as bad as the optimal “offline” ordering.
Skip Lists

- **Randomized** data structure for dictionary ADT
- A hierarchy of ordered linked lists
- A **skip list** for a set $S$ of items is a series of lists $S_0, S_1, \ldots, S_h$ such that:
  - Each list $S_i$ contains the special keys $-\infty$ and $+\infty$
  - List $S_0$ contains the keys of $S$ in non-decreasing order
  - Each list is a subsequence of the previous one, i.e., $S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h$
  - List $S_h$ contains only the two special keys

```
S_0
S_1
S_2
S_3
+\infty
+\infty
```

Search in Skip Lists

- **skip-search**($L$, $k$)
  - $L$: A skip list, $k$: a key
  1. $p \leftarrow$ topmost left position of $L$
  2. $S \leftarrow$ stack of positions, initially containing $p$
  3. **while** $below(p) \neq null$ **do**
     4. $p \leftarrow below(p)$
     5. **while** $key(after(p)) < k$ **do**
        6. $p \leftarrow after(p)$
     7. push $p$ onto $S$
  8. return $S$

- $S$ contains positions of the largest key **less than** $k$ at each level.
- $after(top(S))$ will have key $k$, iff $k$ is in $L$.
- **drop down**: $p \leftarrow below(p)$
- **scan forward**: $p \leftarrow after(p)$
Search in Skip Lists

Example: Skip-Search(\(S, 87\))

\[
\begin{align*}
S_0 & \rightarrow 37 \\
S_1 & \rightarrow 44, 61, 69, 79, 81, 94, 97, 94, 94 \\
S_2 & \rightarrow 23, 37, 44, 61, 69, 79, 81, 94, 94, 97, 94, 94
\end{align*}
\]

Insert in Skip Lists

- Skip-Insert(\(S, k, v\))
  - Randomly compute the height of new item: repeatedly toss a coin until you get tails, let \(i\) be the number of times the coin came up heads
  - Search for \(k\) in the skip list and find the positions \(p_0, p_1, \cdots, p_i\) of the items with largest key less than \(k\) in each list \(S_0, S_1, \cdots, S_i\) (by performing Skip-Search(\(S, k\)))
  - Insert item \((k, v)\) into list \(S_j\) after position \(p_j\) for \(0 \leq j \leq i\) (a tower of height \(i\))

Example: Skip-Insert(\(S, 52, v\))

\[
\begin{align*}
S_0 & \rightarrow 37 \\
S_1 & \rightarrow 44, 61, 69, 79, 81, 94, 94, 97, 94, 94 \\
S_2 & \rightarrow 23, 37, 44, 61, 69, 79, 81, 94, 94, 97, 94, 94
\end{align*}
\]
**Insert in Skip Lists**

Example: \( \text{Skip-Insert}(S, 100, v) \)

\[
\begin{align*}
S_0 & \quad \text{---} \quad \text{---} \\
S_1 & \quad \text{---} \quad 65 \\
S_2 & \quad \text{---} \quad \text{---} \\
\end{align*}
\]

**Delete in Skip Lists**

- **Skip-Delete(\( S, k \))**
  - Search for \( k \) in the skip list and find all the positions \( p_0, p_1, \ldots, p_i \) of the items with the largest key smaller than \( k \), where \( p_j \) is in list \( S_j \).
    - (this is the same as Skip-Search)
  - For each \( i \), if \( \text{key}(\text{after}(p_i)) = k \), then remove \( \text{after}(p_i) \) from list \( S_i \).
  - Remove all but one of the lists \( S_i \) that contain only the two special keys.

**Delete in Skip Lists**

Example: \( \text{Skip-Delete}(S, 65) \)
Summary of Skip Lists

- Expected space usage: $O(n)$
- Expected height: $O(\log n)$
  - A skip list with $n$ items has height at most $3 \log n$ with probability at least $1 - 1/n^2$
- Skip-Search: $O(\log n)$ expected time
- Skip-Insert: $O(\log n)$ expected time
- Skip-Delete: $O(\log n)$ expected time
- Skip lists are fast and simple to implement in practice