Module 5: Dictionaries II

CS 240 - Data Structures and Data Management

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Dictionary ADT: Review

A dictionary is a collection of key-value pairs (KVPs), supporting operations search, insert, and delete.

Realizations

- Unordered array or linked list: $\Theta(1)$ insert, $\Theta(n)$ search and delete
- Ordered array: $\Theta(\log n)$ search, $\Theta(n)$ insert and delete
- Balanced search trees (AVL trees): $\Theta(\log n)$ search, insert, and delete

Self-Organizing Search

- Unordered linked list
  - search: $\Theta(n)$, insert: $\Theta(1)$, delete: $\Theta(1)$ (after a search)
  - Linear search to find an item in the list
  - Is there a more useful ordering?
  - No: if items are accessed equally likely
  - Yes: otherwise (we have a probability distribution for items)
- Optimal static ordering: sorting items by their probabilities of access in non-increasing order minimizes the expected cost of Search.
- Proof Idea: For any other ordering, exchanging two items that are out-of-order according to their access probabilities makes the total cost decrease.

Optimal Static Ordering

A list of elements ordered by non-increasing probability of access has minimum expected access cost

- $L = \langle x_1, x_2, \ldots, x_n \rangle$
  - Expected access cost in $L$ is
    \[
    E(L) = \sum_{i=1}^{n} P(x_i) T(x_i) = \sum_{i=1}^{n} P(x_i) \cdot i
    \]
  - $P(x_i)$ - access probability for $x_i$
  - $T(x_i)$ - position of $x_i$ in $L$
- Example
  - $P(a) = 0.3$ $P(b) = 0.5$ $P(c) = 0.2$
  - $L = \langle a, b, c \rangle$
  - $E(L) = 0.3 + 0.5 \times 2 + 0.2 \times 3 = 1.9$
  - $L = \langle b, a, c \rangle$
  - $E(L) = 0.5 + 0.3 \times 2 + 0.2 \times 3 = 1.7$
Optimal Static Ordering

A list of elements ordered by non-increasing probability of access has minimum expected access cost

Proof by Contradiction

\( L = \langle x_1, \ldots, x_k, x_{k+1}, \ldots, x_n \rangle \)

Suppose the access cost of \( L \) is optimal and there is \( k \) such that 
\[ P(x_k) < P(x_{k+1}) \]

\[ E(L) = P(x_k) \cdot k + P(x_{k+1}) \cdot (k + 1) + \sum_{i \neq k, k+1} P(x_i) \cdot i \]

Create another list \( L' \) by swapping \( x_k \) and \( x_{k+1} \).

\( L' = \langle x_1, \ldots, x_{k+1}, x_k, \ldots, x_n \rangle \)

\[ E(L') = P(x_{k+1}) \cdot k + P(x_k) \cdot (k + 1) + \sum_{i \neq k, k+1} P(x_i) \cdot i \]

\[ E(L') - E(L) = P(x_k) - P(x_{k+1}) < 0 \Rightarrow E(L') < E(L) \]

Contradiction

Dynamic Ordering

- What if we do not know the access probabilities ahead of time?
- **Move-To-Front (MTF):** Upon a successful search, move the accessed item to the front of the list
- **Transpose:** Upon a successful search, swap the accessed item with the item immediately preceding it

Performance of dynamic ordering:

- Both can be implemented in arrays or linked lists.
- Transpose does not adapt quickly to changing access patterns.
- MTF Works well in practice.
- Theoretically MTF is “competitive”:
  No more than twice as bad as the optimal “offline” ordering.

Skip Lists

- **Randomized** data structure for dictionary ADT
- A hierarchy of ordered linked lists
- A skip list for a set \( S \) of items is a series of lists \( S_0, S_1, \cdots, S_h \) such that:
  - Each list \( S_i \) contains the special keys \(-\infty\) and \(+\infty\)
  - List \( S_0 \) contains the keys of \( S \) in non-decreasing order
  - Each list is a subsequence of the previous one, i.e., \( S_0 \supseteq S_1 \supseteq \cdots \supseteq S_h \)
  - List \( S_h \) contains only the two special keys

![Skip List Diagram](image)
Search in Skip Lists

skip-search(L, k)
L: A skip list, k: a key
1. p ← topmost left position of L
2. S ← stack of positions, initially containing p
3. while below(p) ≠ null do
   4. p ← below(p)
   5. while key(after(p)) < k do
      6. p ← after(p)
      7. push p onto S
8. return S

- S contains positions of the largest key less than k at each level.
- after(top(S)) will have key k, iff k is in L.
- drop down: p ← below(p)
- scan forward: p ← after(p)

Insert in Skip Lists

Skip-Insert(S, k, v)
- Randomly compute the height of new item: repeatedly toss a coin until you get tails, let i be the number of times the coin came up heads
- Search for k in the skip list and find the positions p0, p1, ..., pi of the items with largest key less than k in each list S0, S1, ..., Si (by performing Skip-Search(S, k))
- Insert item (k, v) into list Sj after position pj for 0 ≤ j ≤ i (a tower of height i)
Delete in Skip Lists

Example: Skip-Delete(S, 65)

\[ S_1 \quad \infty \quad 65 \quad \infty \]
\[ S_0 \quad \infty \quad 37 \quad 65 \quad 83 \quad 94 \quad \infty \]
\[ S_2 \quad \infty \quad 23 \quad 37 \quad 44 \quad 65 \quad 69 \quad 79 \quad 94 \quad 83 \quad 87 \]
\[ S_3 \quad \infty \quad 23 \quad 37 \quad 44 \quad 65 \quad 69 \quad 79 \quad 94 \quad 83 \quad 87 \]