Module 8: Data Structures for Multi-Dimensional Data

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Multi-Dimensional Data

- Various applications
  - Attributes of a product (laptop: price, screen size, processor speed, RAM, hard drive, ⋅⋅⋅)
  - Attributes of an employee (name, age, salary, ⋅⋅⋅)

- Dictionary for multi-dimensional data
  A collection of $d$-dimensional items
  Each item has $d$ aspects (coordinates): $(x_0, x_1, ⋅⋅⋅, x_{d-1})$
  Operations: insert, delete, range-search query

- (Orthogonal) Range-search query: specify a range (interval) for certain aspects, and find all the items whose aspects fall within given ranges.
  Example: laptops with screen size between 11 and 13 inches, RAM between 8 and 16 GB, price between 1,500 and 2,000 CAD
Multi-Dimensional Data

- Each item has \(d\) aspects (coordinates): \((x_0, x_1, \cdots, x_{d-1})\)
- Aspect values \((x_i)\) are numbers
- Each item corresponds to a point in \(d\)-dimensional space
- We concentrate on \(d = 2\), i.e., points in Euclidean plane

![Graphical representation of multi-dimensional data with price and processor speed dimensions.](image)

One-Dimensional Range Search

- **First solution**: ordered arrays
  - Running time:
  - Problem: does not generalize to higher dimensions
- **Second solution**: balanced BST (e.g., AVL tree)

\[
\begin{align*}
\text{BST-RangeSearch}(T, k_1, k_2) \\
T: \text{A balanced search tree, } k_1, k_2: \text{search keys} \\
&\text{Report keys in } T \text{ that are in range } [k_1, k_2] \\
1. &\text{if } T = \text{nil} \text{ then return} \\
2. &\text{if } \text{key}(T) < k_1 \text{ then} \\
3. &\text{BST-RangeSearch}(T.\text{right}, k_1, k_2) \\
4. &\text{if } \text{key}(T) > k_2 \text{ then} \\
5. &\text{BST-RangeSearch}(T.\text{left}, k_1, k_2) \\
6. &\text{if } k_1 \leq \text{key}(T) \leq k_2 \text{ then} \\
7. &\text{BST-RangeSearch}(T.\text{left}, k_1, k_2) \\
8. &\text{report } \text{key}(T) \\
9. &\text{BST-RangeSearch}(T.\text{right}, k_1, k_2)
\end{align*}
\]
Range Search example

\( BST-RangeSearch(T, 30, 65) \)

Nodes either on boundary, inside, or outside.

Note: Not every boundary node is returned.

One-Dimensional Range Search

- \( P_1 \): path from the root to a leaf that goes right if \( k < k_1 \) and left otherwise
- \( P_2 \): path from the root to a leaf that goes left if \( k > k_2 \) and right otherwise

Partition nodes of \( T \) into three groups:

- **boundary nodes**: nodes in \( P_1 \) or \( P_2 \)
- **inside nodes**: non-boundary nodes that belong to either (a subtree rooted at a right child of a node of \( P_1 \)) or (a subtree rooted at a left child of a node of \( P_2 \))
- **outside nodes**: non-boundary nodes that belong to either (a subtree rooted at a left child of a node of \( P_1 \)) or (a subtree rooted at a right child of a node of \( P_2 \))

- \( k \): number of reported items

Nodes visited during the search:

- \( O(\log n) \) boundary nodes
- \( O(k) \) inside nodes
- No outside nodes

Running time \( O(\log n + k) \)
2-Dimensional Range Search

- Each item has 2 aspects (coordinates): \((x_i, y_i)\)
- Each item corresponds to a point in Euclidean plane
- Options for implementing \(d\)-dimensional dictionaries:
  - Reduce to one-dimensional dictionary: combine the \(d\)-dimensional key into one key
    Problem: Range search on one aspect is not straightforward
  - Use several dictionaries: one for each dimension
    Problem: inefficient, wastes space
  - Partition trees
    - A tree with \(n\) leaves, each leaf corresponds to an item
    - Each internal node corresponds to a region
    - quadtrees, \(kd\)-trees
  - multi-dimensional range trees

Quadtrees

- We have \(n\) points \(P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}\) in the plane
- How to build a quadtree on \(P\):
  - Find a square \(R\) that contains all the points of \(P\) (We can compute minimum and maximum \(x\) and \(y\) values among \(n\) points)
  - Root of the quadtree corresponds to \(R\)
  - Split: Partition \(R\) into four equal subsquares (quadrants), each correspond to a child of \(R\)
  - Recursively repeat this process for any node that contains more than one point
  - Points on split lines belong to left/bottom side
  - Each leaf stores (at most) one point
  - We can delete a leaf that does not contain any point
Quadtrees

- Example: We have 13 points \( P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{12}, y_{12})\} \) in the plane.

Quadtree Operations

- **Search**: Analogous to binary search trees
- **Insert**:
  - Search for the point
  - Split the leaf if there are two points
- **Delete**:
  - Search for the point
  - Remove the point
  - If its parent has only one child left, delete that child and continue the process toward the root.
Quadtree: Range Search

\[ QTree-RangeSearch(T, R) \]
\[ T: A \text{ quadtree node, } R: \text{ Query rectangle} \]
1. \text{ if } (T \text{ is a leaf}) \text{ then}
2. \quad \text{ if } (T.\text{point} \in R) \text{ then}
3. \quad \quad \text{ report } T.\text{point}
4. \quad \text{ for each child } C \text{ of } T \text{ do}
5. \quad \quad \text{ if } C.\text{region} \cap R \neq \emptyset \text{ then}
6. \quad \quad \quad QTree-RangeSearch(C, R)

- **spread factor** of points \( P \): \( \beta(P) = \frac{d_{\text{max}}}{d_{\text{min}}} \)
- \( d_{\text{max}}(d_{\text{min}}) \): maximum (minimum) distance between two points in \( P \)
- **height** of quadtree: \( h \in \Theta(\log_2 \frac{d_{\text{max}}}{d_{\text{min}}}) \)
- Complexity to build initial tree: \( \Theta(nh) \)
- Complexity of range search: \( \Theta(nh) \) even if the answer is \( \emptyset \)

Quadtree Conclusion

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (usually the boundary box is padded to get a power of two).
- Space wasteful
- Major drawback: can have very large height for certain nonuniform distributions of points
- Easily generates to higher dimensions (octrees, etc.).
kd-trees

- We have $n$ points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane.
- Quadtrees split square into quadrants regardless of where points actually lie.
- kd-tree idea: Split the points into two (roughly) equal subsets.
- How to build a kd-tree on $P$:
  - Split $P$ into two equal subsets using a vertical line.
  - Split each of the two subsets into two equal pieces using horizontal lines.
  - Continue splitting, alternating vertical and horizontal lines, until every point is in a separate region.
- More details:
  - Initially, we sort the $n$ points according to their $x$-coordinates.
  - The root of the tree is the point with median $x$ coordinate (index $\lfloor n/2 \rfloor$ in the sorted list).
  - All other points with $x$ coordinate less than or equal to this go into the left subtree; points with larger $x$-coordinate go in the right subtree.
  - At alternating levels, we sort and split according to $y$-coordinates instead.

**Complexity**: $\Theta(n \log n)$, **height of the tree**: $\Theta(\log n)$

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kd-trees

- kd-tree idea: Split the points into two (roughly) equal subsets.
- A **balanced** binary tree:
  
  - $p_4$
  - $p_3$  $p_9$
  - $p_8$
  - $p_1$
  - $p_0$
  - $p_2$  $p_6$  $p_5$
  - $p_7$
kd-tree: Range Search

kd-rangeSearch(T, R, split[←‘x’])

T: A kd-tree node, R: Query rectangle

1. if T is empty then return
2. if T.point ∈ R then
3. report T.point
4. for each child C of T do
5. if C.region ∩ R ≠ ∅ then
6. kd-rangeSearch(C, R)
7. if split = ‘x’ then
8. if T.point.x ≥ R.leftSide then
9. kd-rangeSearch(T.left, R, ‘y’)
10. if T.point.x < R.rightSide then
11. kd-rangeSearch(T.right, R, ‘y’)
12. if split = ‘y’ then
13. if T.point.y ≥ R.bottomSide then
14. kd-rangeSearch(T.left, R, ‘x’)
15. if T.point.y < R.topSide then
16. kd-rangeSearch(T.right, R, ‘x’)

kd-tree: Range Search Complexity

- The complexity is $O(k + U)$ where $k$ is the number of keys reported and $U$ is the number of regions we go to but unsuccessfully.
- $U$ corresponds to the number of regions which intersect but are not fully in $R$.
- Those regions have to intersect one of the four sides of $R$.
- $Q(n)$: Maximum number of regions in a kd-tree with $n$ points that intersect a vertical (horizontal) line.
- $Q(n)$ satisfies the following recurrence relation:
  
  $$Q(n) = 2Q(n/4) + O(1)$$

- It solves to $Q(n) = O(\sqrt{n})$.
- Therefore, the complexity of range search in kd-trees is $O(k + \sqrt{n})$. 
kd-tree: Higher Dimensions

- **kd-trees for** $d$-**dimensional space**
  - At the root the point set is partitioned based on the first coordinate
  - At the children of the root the partition is based on the second coordinate
  - At depth $d - 1$ the partition is based on the last coordinate
  - At depth $d$ we start all over again, partitioning on first coordinate

- **Storage**: $O(n)$
- **Construction time**: $O(n \log n)$
- **Range query time**: $O(n^{1-1/d} + k)$

(Note: $d$ is considered to be a constant.)

Range Trees

- We have $n$ points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane
- A range tree is a **tree of trees** (a *multi-level* data structure)
- How to **build** a range tree on $P$:
  - Build a balanced binary search tree $\tau$ determined by the $x$-coordinates of the $n$ points
  - For every node $v \in \tau$, build a balanced binary search tree $\tau_{assoc}(v)$ (associated structure of $\tau$) determined by the $y$-coordinates of the nodes in the subtree of $\tau$ with root node $v$
Range Trees: Operations

- **Search**: trivially as in a binary search tree
- **Insert**: insert point in $\tau$ by $x$-coordinate
  - From inserted leaf, walk back up to the root and insert the point in all associated trees $\mathcal{T}_{assoc}(v)$ of nodes $v$ on path to the root
- **Delete**: analogous to insertion
- **Note**: re-balancing is a problem!
Range Trees: Range Search

- A two stage process
- To perform a range search query $R = [x_1, x_2] \times [y_1, y_2]$:
  - Perform a range search (on the $x$-coordinates) for the interval $[x_1, x_2]$ in $\tau$ ($\text{BST-RangeSearch}(\tau, x_1, x_2)$)
  - For every outside node, do nothing.
  - For every “top” inside node $\nu$, perform a range search (on the $y$-coordinates) for the interval $[y_1, y_2]$ in $\tau_{\text{assoc}}(\nu)$. During the range search of $\tau_{\text{assoc}}(\nu)$, do not check any $x$-coordinates (they are all within range).
  - For every boundary node, test to see if the corresponding point is within the region $R$.

- Running time: $O(k + \log^2 n)$
- Range tree space usage: $O(n \log n)$

Range Trees: Higher Dimensions

- Range trees for $d$-dimensional space
- Space/time trade-off
  - Storage: $O(n (\log n)^{d-1})$  
    kd-trees: $O(n)$
  - Construction time: $O(n (\log n)^{d-1})$  
    kd-trees: $O(n \log n)$
  - Range query time: $O((\log n)^d + k)$  
    kd-trees: $O(n^{1-1/d} + k)$

(Note: $d$ is considered to be a constant.)