Module 8: Data Structures for Multi-Dimensional Data

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Multi-Dimensional Data

- Various applications
  - Attributes of a product (laptop: price, screen size, processor speed, RAM, hard drive,...)
  - Attributes of an employee (name, age, salary,...)
- Dictionary for multi-dimensional data
  A collection of $d$-dimensional items
  Each item has $d$ aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
  Operations: insert, delete, **range-search query**

- (Orthogonal) Range-search query: specify a range (interval) for certain aspects, and find all the items whose aspects fall within given ranges.
  Example: laptops with screen size between 11 and 13 inches, RAM between 8 and 16 GB, price between 1,500 and 2,000 CAD

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**Diagram:**

- Each item has $d$ aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
- Aspect values ($x_i$) are numbers
- Each item corresponds to a point in $d$-dimensional space
- We concentrate on $d = 2$, i.e., points in Euclidean plane
One-Dimensional Range Search

- **First solution**: ordered arrays
  - Running time:
  - Problem: does not generalize to higher dimensions
- **Second solution**: balanced BST (e.g., AVL tree)

\[
\text{BST-RangeSearch}(T, k_1, k_2)
\]

\[T: \text{A balanced search tree, } k_1, k_2: \text{search keys}\]

Report keys in \(T\) that are in range \([k_1, k_2]\)

1. if \(T = \text{nil}\) then return
2. if \(\text{key}(T) < k_1\) then
3. \(\text{BST-RangeSearch}(T.\text{right}, k_1, k_2)\)
4. if \(\text{key}(T) > k_2\) then
5. \(\text{BST-RangeSearch}(T.\text{left}, k_1, k_2)\)
6. if \(k_1 \leq \text{key}(T) \leq k_2\) then
7. \(\text{BST-RangeSearch}(T.\text{left}, k_1, k_2)\)
8. \(\text{report } \text{key}(T)\)
9. \(\text{BST-RangeSearch}(T.\text{right}, k_1, k_2)\)

Range Search example

\[
\text{BST-RangeSearch}(T, 30, 65)
\]

Nodes either on boundary, inside, or outside.

Note: Not every boundary node is returned.

One-Dimensional Range Search

- \(P_1\): path from the root to a leaf that goes right if \(k < k_1\) and left otherwise
- \(P_2\): path from the root to a leaf that goes left if \(k > k_2\) and right otherwise
- Partition nodes of \(T\) into three groups:
  - **boundary nodes**: nodes in \(P_1\) or \(P_2\)
  - **inside nodes**: non-boundary nodes that belong to either (a subtree rooted at a right child of a node of \(P_1\)) or (a subtree rooted at a left child of a node of \(P_2\))
  - **outside nodes**: non-boundary nodes that belong to either (a subtree rooted at a left child of a node of \(P_1\)) or (a subtree rooted at a right child of a node of \(P_2\))
- \(k\): number of reported items
- Nodes visited during the search:
  - \(O(\log n)\) boundary nodes
  - \(O(k)\) inside nodes
  - No outside nodes
- Running time \(O(\log n + k)\)
2-Dimensional Range Search

- Each item has 2 aspects (coordinates): \((x_i, y_i)\)
- Each item corresponds to a point in Euclidean plane
- Options for implementing \(d\)-dimensional dictionaries:
  - Reduce to one-dimensional dictionary: combine the \(d\)-dimensional key into one key
    - Problem: Range search on one aspect is not straightforward
  - Use several dictionaries: one for each dimension
    - Problem: inefficient, wastes space
  - Partition trees
    - A tree with \(n\) leaves, each leaf corresponds to an item
    - Each internal node corresponds to a region
    - quadtrees, kd-trees
  - multi-dimensional range trees

Quadtrees

- We have \(n\) points \(P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}\) in the plane
- How to build a quadtree on \(P\):
  - Find a square \(R\) that contains all the points of \(P\) (We can compute minimum and maximum \(x\) and \(y\) values among \(n\) points)
  - Root of the quadtree corresponds to \(R\)
  - Split: Partition \(R\) into four equal subsquares (quadrants), each correspond to a child of \(R\)
  - Recursively repeat this process for any node that contains more than one point
  - Points on split lines belong to left/bottom side
  - Each leaf stores (at most) one point
  - We can delete a leaf that does not contain any point

Example: We have 13 points \(P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{12}, y_{12})\}\) in the plane

```
  *  *
  *  *
  *  *
  *  *
  *  *
  *  *
```
Quadtree Operations

- **Search**: Analogous to binary search trees
- **Insert**:
  - Search for the point
  - Split the leaf if there are two points
- **Delete**:
  - Search for the point
  - Remove the point
  - If its parent has only one child left, delete that child and continue the process toward the root.

Quadtree: Range Search

\[ \text{QTree-RangeSearch}(T, R) \]
\[ T: \text{A quadtree node}, \ R: \text{Query rectangle} \]
1. if (\( T \) is a leaf) then
2. if (\( T.\text{point} \in R \)) then
3. report \( T.\text{point} \)
4. for each child \( C \) of \( T \) do
5. if \( C.\text{region} \cap R \neq \emptyset \) then
6. \( \text{QTree-RangeSearch}(C, R) \)

- **spread factor** of points \( P \): \( \beta(P) = \frac{d_{\text{max}}}{d_{\text{min}}} \)
- \( d_{\text{max}}(d_{\text{min}}) \): maximum (minimum) distance between two points in \( P \)
- **height** of quadtree: \( h \in \Theta(\log_2 \frac{d_{\text{max}}}{d_{\text{min}}}) \)
- Complexity to build initial tree: \( \Theta(nh) \)
- Complexity of range search: \( \Theta(nh) \) even if the answer is \( \emptyset \)

Quadtree Conclusion

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (usually the boundary box is padded to get a power of two).
- Space wasteful
- Major drawback: can have very large height for certain nonuniform distributions of points
- Easily generates to higher dimensions (octrees, etc.).
kd-trees
- We have \( n \) points \( P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\} \) in the plane
- Quadtrees split square into quadrants regardless of where points actually lie
- kd-tree idea: Split the points into two (roughly) equal subsets
- How to build a kd-tree on \( P \):
  - Split \( P \) into two equal subsets using a vertical line
  - Split each of the two subsets into two equal pieces using horizontal lines
  - Continue splitting, alternating vertical and horizontal lines, until every point is in a separate region
- More details:
  - Initially, we sort the \( n \) points according to their \( x \)-coordinates.
  - The root of the tree is the point with median \( x \) coordinate (index \( \lfloor n/2 \rfloor \) in the sorted list)
  - All other points with \( x \) coordinate less than or equal to this go into the left subtree; points with larger \( x \)-coordinate go in the right subtree.
  - At alternating levels, we sort and split according to \( y \)-coordinates instead.
- Complexity: \( \Theta(n \log n) \), height of the tree: \( \Theta(\log n) \)

**kd-tree: Range Search**

```
k-rangeSearch(T, R, split['x'])
T: A kd-tree node, R: Query rectangle
1. if T is empty then return
2. if T.point \( \in \) R then
3. report T.point
4. for each child C of T do
5. if C.region \( \cap \) R \( \neq \) \( \emptyset \) then
6. k-rangeSearch(C, R)
7. if split = 'x' then
8. if T.point.x \( \geq \) R.leftSide then
9. k-rangeSearch(T.left, R, 'y')
10. if T.point.x < R.rightSide then
11. k-rangeSearch(T.right, R, 'y')
12. if split = 'y' then
13. if T.point.y \( \geq \) R.bottomSide then
14. k-rangeSearch(T.left, R, 'x')
15. if T.point.y < R.topSide then
16. k-rangeSearch(T.right, R, 'x')
```
kd-tree: Range Search Complexity

- The complexity is $O(k + U)$ where $k$ is the number of keys reported and $U$ is the number of regions we go to but unsuccessfully.
- $U$ corresponds to the number of regions which intersect but are not fully in $R$.
- Those regions have to intersect one of the four sides of $R$.
- $Q(n)$: Maximum number of regions in a kd-tree with $n$ points that intersect a vertical (horizontal) line.
- $Q(n)$ satisfies the following recurrence relation:
  \[
  Q(n) = 2Q(n/4) + O(1)
  \]
- It solves to $Q(n) = O(\sqrt{n})$.
- Therefore, the complexity of range search in kd-trees is $O(k + \sqrt{n})$.

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kd-tree: Higher Dimensions

- kd-trees for $d$-dimensional space
  - At the root the point set is partitioned based on the first coordinate.
  - At the children of the root the partition is based on the second coordinate.
  - At depth $d - 1$ the partition is based on the last coordinate.
  - At depth $d$ we start all over again, partitioning on first coordinate.
- Storage: $O(n)$
- Construction time: $O(n \log n)$
- Range query time: $O(n^{1-1/d} + k)$
  (Note: $d$ is considered to be a constant.)

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Range Trees

- We have $n$ points $P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$ in the plane.
- A range tree is a tree of trees (a multi-level data structure).
- How to build a range tree on $P$:
  - Build a balanced binary search tree $\tau$ determined by the $x$-coordinates of the $n$ points.
  - For every node $v \in \tau$, build a balanced binary search tree $\tau_{assoc}(v)$ (associated structure of $\tau$) determined by the $y$-coordinates of the nodes in the subtree of $\tau$ with root node $v$.

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### Range Tree Structure

![Diagram of Range Tree Structure](image)

- **Binary search tree on** $x$-coordinates
- **Binary search tree on** $y$-coordinates
- **$T_{assoc}(v)$**
- **$P(\nu)$**

### Range Trees: Operations

- **Search**: trivially as in a binary search tree
- **Insert**: insert point in $\tau$ by $x$-coordinate
- **From inserted leaf, walk back up to the root and insert the point in all associated trees $T_{assoc}(v)$ of nodes $v$ on path to the root**
- **Delete**: analogous to insertion
- **Note**: re-balancing is a problem!

### Range Trees: Range Search

- **A two stage process**
- **To perform a range search query $R = [x_1, x_2] \times [y_1, y_2]$**:
  - Perform a range search (on the $x$-coordinates) for the interval $[x_1, x_2]$ in $\tau$ ($BST$-RangeSearch($\tau, x_1, x_2$))
  - For every **outside node**, do nothing.
  - For every **“top” inside node** $v$, perform a range search (on the $y$-coordinates) for the interval $[y_1, y_2]$ in $T_{assoc}(v)$. During the range search of $T_{assoc}(v)$, do not check any $x$-coordinates (they are all within range).
  - For every **boundary node**, test to see if the corresponding point is within the region $R$.
- **Running time**: $O(k + \log^2 n)$
- **Range tree space usage**: $O(n \log n)$
Range Trees: Higher Dimensions

- Range trees for \(d\)-dimensional space
- Space/time trade-off
  - **Storage:** \(O(n (\log n)^{d-1})\)
  - **Construction time:** \(O(n (\log n)^{d-1})\)
  - **Range query time:** \(O((\log n)^d + k)\)

(Note: \(d\) is considered to be a constant.)