Module 8: Data Structures for Multi-Dimensional Data

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Multi-Dimensional Data

- Various applications
  - Attributes of a product (laptop: price, screen size, processor speed, RAM, hard drive, ···)
  - Attributes of an employee (name, age, salary, ···)

- Dictionary for multi-dimensional data
  A collection of $d$-dimensional items
  Each item has $d$ aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
  Operations: insert, delete, range-search query

- (Orthogonal) Range-search query: specify a range (interval) for certain aspects, and find all the items whose aspects fall within given ranges.
  Example: laptops with screen size between 11 and 13 inches, RAM between 8 and 16 GB, price between 1,500 and 2,000 CAD
Multi-Dimensional Data

- Each item has \textit{d aspects} (coordinates): \((x_0, x_1, \cdots, x_{d-1})\)
- Aspect values \((x_i)\) are numbers
- Each item corresponds to a point in \(d\)-dimensional space
- We concentrate on \(d = 2\), i.e., points in Euclidean plane

![Graph](https://via.placeholder.com/150)

- item: ordered pair \((x, y) \in \mathbb{R} \times \mathbb{R}\)
- range-search query \((1350 \leq x \leq 1550, 700 \leq y \leq 1100)\)
- \((1200, 1000)\)
One-Dimensional Range Search

- **First solution**: ordered arrays
  - Running time:
  - Problem: does not generalize to higher dimensions

- **Second solution**: balanced BST (e.g., AVL tree)

```latex
\begin{align*}
  BST\text{-RangeSearch}(T, k_1, k_2) \\
  T: \text{A balanced search tree, } k_1, k_2: \text{search keys} \\
  \text{Report keys in } T \text{ that are in range } [k_1, k_2] \\
  1. \text{ if } T = \text{nil} \text{ then return} \\
  2. \text{ if } key(T) < k_1 \text{ then} \\
     3. \quad \text{BST-RangeSearch}(T.\text{right}, k_1, k_2) \\
  4. \text{ if } key(T) > k_2 \text{ then} \\
     5. \quad \text{BST-RangeSearch}(T.\text{left}, k_1, k_2) \\
  6. \text{ if } k_1 \leq key(T) \leq k_2 \text{ then} \\
     7. \quad \text{BST-RangeSearch}(T.\text{left}, k_1, k_2) \\
     8. \quad \text{report } key(T) \\
     9. \quad \text{BST-RangeSearch}(T.\text{right}, k_1, k_2)
\end{align*}
```
Range Search example

\( BST\text{-RangeSearch}(T, 30, 65) \)
Nodes either on boundary, inside, or outside.

Note: Not every boundary node is returned.
One-Dimensional Range Search

- \( P_1 \): path from the root to a leaf that goes right if \( k < k_1 \) and left otherwise
- \( P_2 \): path from the root to a leaf that goes left if \( k > k_2 \) and right otherwise
- Partition nodes of \( T \) into three groups:
  1. **boundary nodes**: nodes in \( P_1 \) or \( P_2 \)
  2. **inside nodes**: non-boundary nodes that belong to either (a subtree rooted at a right child of a node of \( P_1 \)) or (a subtree rooted at a left child of a node of \( P_2 \))
  3. **outside nodes**: non-boundary nodes that belong to either (a subtree rooted at a left child of a node of \( P_1 \)) or (a subtree rooted at a right child of a node of \( P_2 \))
- \( k \): number of reported items
- Nodes visited during the search:
  - \( O(\log n) \) boundary nodes
  - \( O(k) \) inside nodes
  - No outside nodes
- Running time \( O(\log n + k) \)
2-Dimensional Range Search

- Each item has 2 **aspects** (coordinates): \((x_i, y_i)\)
- Each item corresponds to a point in Euclidean plane
- Options for implementing \(d\)-dimensional dictionaries:
  - Reduce to one-dimensional dictionary: combine the \(d\)-dimensional key into one key
    Problem: Range search on one aspect is not straightforward
  - Use several dictionaries: one for each dimension
    Problem: inefficient, wastes space
  - **Partition trees**
    - A tree with \(n\) leaves, each leaf corresponds to an item
    - Each internal node corresponds to a region
    - **quadtrees, kd-trees**
  - multi-dimensional **range trees**
Quadtrees

- We have \( n \) points \( P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\} \) in the plane.

- How to **build** a quadtree on \( P \):
  - Find a square \( R \) that contains all the points of \( P \) (We can compute minimum and maximum \( x \) and \( y \) values among \( n \) points).
  - Root of the quadtree corresponds to \( R \).
  - **Split**: Partition \( R \) into four equal subsquares (**quadrants**), each correspond to a child of \( R \).
  - Recursively repeat this process for any node that contains more than one point.
  - Points on split lines belong to left/bottom side.
  - Each leaf stores (at most) one point.
  - We can delete a leaf that does not contain any point.
Quadtrees

Example: We have 13 points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{12}, y_{12})\}$ in the plane
Quadtree Operations

- **Search**: Analogous to binary search trees
- **Insert**:
  - Search for the point
  - Split the leaf if there are two points
- **Delete**:
  - Search for the point
  - Remove the point
  - If its parent has only one child left, delete that child and continue the process toward the root.
Quadtree: Range Search

\[
\text{QTree-RangeSearch}(T, R)
\]

- **\( T \):** A quadtree node, **\( R \):** Query rectangle
- 1. **if** (\( T \) is a leaf) **then**
- 2. **if** (\( T \).point \( \in \) \( R \)) **then**
- 3. report \( T \).point
- 4. **for** each child \( C \) of \( T \) **do**
- 5. **if** \( C \).region \( \cap \) \( R \) \( \neq \) \( \emptyset \) **then**
- 6. \( \text{QTree-RangeSearch}(C, R) \)

- **spread factor** of points \( P \): \( \beta(P) = \frac{d_{\text{max}}}{d_{\text{min}}} \)
- \( d_{\text{max}}(d_{\text{min}}) \): maximum (minimum) distance between two points in \( P \)
- **height** of quadtree: \( h \in \Theta(\log_2 \frac{d_{\text{max}}}{d_{\text{min}}}) \)
- Complexity to build initial tree: \( \Theta(nh) \)
- Complexity of range search: \( \Theta(nh) \) even if the answer is \( \emptyset \)
Quadtree Conclusion

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (usually the boundary box is padded to get a power of two).
- Space wasteful
- Major drawback: can have very large height for certain nonuniform distributions of points
- Easily generates to higher dimensions (octrees, etc.).
kd-trees

- We have $n$ points $P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$ in the plane.
- Quadtrees split square into quadrants regardless of where points actually lie.
- kd-tree idea: Split the points into two (roughly) equal subsets.
- How to build a kd-tree on $P$:
  - Split $P$ into two equal subsets using a vertical line.
  - Split each of the two subsets into two equal pieces using horizontal lines.
  - Continue splitting, alternating vertical and horizontal lines, until every point is in a separate region.
- More details:
  - Initially, we sort the $n$ points according to their $x$-coordinates.
  - The root of the tree is the point with median $x$ coordinate (index $\lfloor n/2 \rfloor$ in the sorted list).
  - All other points with $x$ coordinate less than or equal to this go into the left subtree; points with larger $x$-coordinate go in the right subtree.
  - At alternating levels, we sort and split according to $y$-coordinates instead.
- **Complexity**: $\Theta(n \log n)$, **height of the tree**: $\Theta(\log n)$.
kd-trees

- **kd-tree idea**: Split the points into two (roughly) equal subsets
- A **balanced** binary tree

```
    p0
     ▼
    p1
     ▼
  p2   p3
     ▼   ▼
 p4   p5   p6   p7
```

Haque, Irvine, Smith  (SCS, UW)
kd-tree: Range Search

kd-rangeSearch( T, R, split[← ‘x’])
T: A kd-tree node, R: Query rectangle
1. if T is empty then return
2. if T.point ∈ R then
3. report T.point
4. for each child C of T do
5. if C.region ∩ R ≠ ∅ then
6. kd-rangeSearch(C, R)
7. if split = ‘x’ then
8. if T.point.x ≥ R.leftSide then
9. kd-rangeSearch(T.left, R, ‘y’)
10. if T.point.x < R.rightSide then
11. kd-rangeSearch(T.right, R, ‘y’)
12. if split = ‘y’ then
13. if T.point.y ≥ R.bottomSide then
14. kd-rangeSearch(T.left, R, ‘x’)
15. if T.point.y < R.topSide then
16. kd-rangeSearch(T.right, R, ‘x’)}
kd-tree: Range Search Complexity

- The complexity is $O(k + U)$ where $k$ is the number of keys reported and $U$ is the number of regions we go to but **unsuccessfully**
- $U$ corresponds to the number of regions which intersect but are not fully in $R$
- Those regions have to intersect one of the four sides of $R$
- $Q(n)$: Maximum number of regions in a kd-tree with $n$ points that intersect a vertical (horizontal) line
- $Q(n)$ satisfies the following recurrence relation:
  \[
  Q(n) = 2Q(n/4) + O(1)
  \]
- It solves to $Q(n) = O(\sqrt{n})$
- Therefore, the complexity of range search in kd-trees is $O(k + \sqrt{n})$
kd-tree: Higher Dimensions

- **kd-trees for** \( d \)-dimensional space
  - At the root the point set is partitioned based on the first coordinate
  - At the children of the root the partition is based on the second coordinate
  - At depth \( d - 1 \) the partition is based on the last coordinate
  - At depth \( d \) we start all over again, partitioning on first coordinate

- **Storage:** \( O(n) \)
- **Construction time:** \( O(n \log n) \)
- **Range query time:** \( O(n^{1-1/d} + k) \)

(Note: \( d \) is considered to be a constant.)
Range Trees

- We have $n$ points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane.
- A range tree is a **tree of trees** (a *multi-level* data structure).
- How to **build** a range tree on $P$:
  - Build a balanced binary search tree $\tau$ determined by the $x$-coordinates of the $n$ points.
  - For every node $v \in \tau$, build a balanced binary search tree $\tau_{assoc}(v)$ (**associated structure of** $\tau$) determined by the $y$-coordinates of the nodes in the subtree of $\tau$ with root node $v$. 
Range Tree Structure

Figure 5.6

A 2-dimensional range tree returns the root of a 2-dimensional range tree $T$ of $P$. As in the previous section, we assume that no two points have the same $x$- or $y$-coordinate. We shall get rid of this assumption in Section 5.5.

Algorithm BUILD2DRTREE($P$)

1. Construct the associated structure: Build a binary search tree $T_{assoc}$ on the set $P_y$ of $y$-coordinates of the points in $P$. Store at the leaves of $T_{assoc}$ not just the $y$-coordinate of the points in $P_y$, but the points themselves.

2. if $P$ contains only one point
   3. then Create a leaf $ν$ storing this point, and make $T_{assoc}$ the associated structure of $ν$.
   4. else Split $P$ into two subsets; one subset $P_{left}$ contains the points with $x$-coordinate less than or equal to $x_{mid}$, the median $x$-coordinate, and the other subset $P_{right}$ contains the points with $x$-coordinate larger than $x_{mid}$.
   5. $ν_{left}$ ← BUILD2DRTREE($P_{left}$)
   6. $ν_{right}$ ← BUILD2DRTREE($P_{right}$)
   7. Create a node $ν$ storing $x_{mid}$, make $ν_{left}$ the left child of $ν$, make $ν_{right}$ the right child of $ν$, and make $T_{assoc}$ the associated structure of $ν$.

8. return $ν$

Note that in the leaves of the associated structures we do not just store the $y$-coordinate of the points but the points themselves. This is important because, when searching the associated structures, we need to report the points and not just the $y$-coordinates.

Lemma 5.6

A range tree on a set of $n$ points in the plane requires $O(n \log n)$ storage.

Proof.

A point $p$ in $P$ is stored only in the associated structure of nodes on the path in $T$ towards the leaf containing $p$. Hence, for all nodes at a given depth of $T$, the number of points stored is at most $O(n \log n)$.
Range Trees: Operations

- **Search**: trivially as in a binary search tree
- **Insert**: insert point in $\tau$ by $x$-coordinate
  - From inserted leaf, walk back up to the root and insert the point in all associated trees $\tau_{assoc}(v)$ of nodes $v$ on path to the root
- **Delete**: analogous to insertion
- **Note**: re-balancing is a problem!
Range Trees: Range Search

- **A two stage process**
- To perform a range search query $R = [x_1, x_2] \times [y_1, y_2]$:
  - Perform a range search (on the $x$-coordinates) for the interval $[x_1, x_2]$ in $\tau$ ($\text{BST-RangeSearch}(\tau, x_1, x_2)$)
  - For every outside node, do nothing.
  - For every “top” inside node $v$, perform a range search (on the $y$-coordinates) for the interval $[y_1, y_2]$ in $\tau_{assoc}(v)$. During the range search of $\tau_{assoc}(v)$, do not check any $x$-coordinates (they are all within range).
  - For every boundary node, test to see if the corresponding point is within the region $R$.
- Running time: $O(k + \log^2 n)$
- Range tree space usage: $O(n \log n)$
Range Trees: Higher Dimensions

- Range trees for $d$-dimensional space
- Space/time trade-off
  - **Storage**: $O(n (\log n)^{d-1})$
  - **Construction time**: $O(n (\log n)^{d-1})$
  - **Range query time**: $O((\log n)^d + k)$

(Note: $d$ is considered to be a constant.)