Module 8: Data Structures for Multi-Dimensional Data

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Multi-Dimensional Data

• Various applications
  ▶ Attributes of a product (laptop: price, screen size, processor speed, RAM, hard drive,···)
  ▶ Attributes of an employee (name, age, salary,···)

• Dictionary for multi-dimensional data
  A collection of $d$-dimensional items
  Each item has $d$ aspects (coordinates): $(x_0, x_1, \cdots, x_{d-1})$
  Operations: insert, delete, range-search query

• (Orthogonal) Range-search query: specify a range (interval) for certain aspects, and find all the items whose aspects fall within given ranges.
  Example: laptops with screen size between 11 and 13 inches, RAM between 8 and 16 GB, price between 1,500 and 2,000 CAD
Multi-Dimensional Data

- Each item has \( d \) aspects (coordinates): \((x_0, x_1, \cdots, x_{d-1})\)
- Aspect values \( (x_i) \) are numbers
- Each item corresponds to a point in \( d \)-dimensional space
- We concentrate on \( d = 2 \), i.e., points in Euclidean plane

![Graph showing multi-dimensional data with price (CAD) on the y-axis and processor speed (MHz) on the x-axis.](image)
One-Dimensional Range Search

- **First solution**: ordered arrays
  - Running time: $O(\log n + k)$, $k$: number of reported items
  - Problem: does not generalize to higher dimensions
- **Second solution**: balanced BST (e.g., AVL tree)

```
BST-RangeSearch(T, k₁, k₂)

T: A balanced search tree, k₁, k₂: search keys
Report keys in T that are in range [k₁, k₂]

1. if T = nil then return
2. if key(T) < k₁ then
   3. BST-RangeSearch(T.right, k₁, k₂)
3. if key(T) > k₂ then
   4. BST-RangeSearch(T.left, k₁, k₂)
5. if k₁ ≤ key(T) ≤ k₂ then
   6. report key(T)
   7. BST-RangeSearch(T.left, k₁, k₂)
   8. BST-RangeSearch(T.right, k₁, k₂)
```
Range Search example

BST-RangeSearch($T, 30, 65$)

Nodes either on boundary, inside, or outside.

Note: Not every boundary node is returned.
Range Search example

$BST$-RangeSearch$(T, 30, 65)$

Nodes either on boundary, inside, or outside.
Range Search example

$BST$-$RangeSearch( T, 30, 65)$$

Nodes either on boundary, inside, or outside.

Note: Not every boundary node is returned.
One-Dimensional Range Search

- \(P_1\): path from the root to a leaf that goes right if \(k < k_1\) and left otherwise
- \(P_2\): path from the root to a leaf that goes left if \(k > k_2\) and right otherwise

Partition nodes of \(T\) into three groups:

1. **boundary nodes**: nodes in \(P_1\) or \(P_2\)
2. **inside nodes**: non-boundary nodes that belong to either (a subtree rooted at a right child of a node of \(P_1\)) or (a subtree rooted at a left child of a node of \(P_2\))
3. **outside nodes**: non-boundary nodes that belong to either (a subtree rooted at a left child of a node of \(P_1\)) or (a subtree rooted at a right child of a node of \(P_2\))
One-Dimensional Range Search

- $P_1$: path from the root to a leaf that goes right if $k < k_1$ and left otherwise
- $P_2$: path from the root to a leaf that goes left if $k > k_2$ and right otherwise
- $k$: number of reported items
- Nodes visited during the search:
  - $O(\log n)$ boundary nodes
  - $O(k)$ inside nodes
  - No outside nodes
- Running time $O(\log n + k)$
2-Dimensional Range Search

- Each item has 2 aspects (coordinates): \((x_i, y_i)\)
- Each item corresponds to a point in Euclidean plane
- Options for implementing \(d\)-dimensional dictionaries:
  - Reduce to one-dimensional dictionary: combine the \(d\)-dimensional key into one key
    Problem: Range search on one aspect is not straightforward
  - Use several dictionaries: one for each dimension
    Problem: inefficient, wastes space
  - Partition trees
    - A tree with \(n\) leaves, each leaf corresponds to an item
    - Each internal node corresponds to a region
    - quadtrees, kd-trees
  - multi-dimensional range trees
Quadtrees

- We have \( n \) points \( P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\} \) in the plane.

- How to build a quadtree on \( P \):
  - Find a square \( R \) that contains all the points of \( P \) (We can compute minimum and maximum \( x \) and \( y \) values among \( n \) points).
  - Root of the quadtree corresponds to \( R \).
  - Split: Partition \( R \) into four equal subsquares (quadrants), each correspond to a child of \( R \).
  - Recursively repeat this process for any node that contains more than one point.
  - Points on split lines belong to left/bottom side.
  - Each leaf stores (at most) one point.
  - We can delete a leaf that does not contain any point.
Example: We have 13 points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{12}, y_{12})\}$ in the plane.
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Quadtrees

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Example: We have 13 points \( P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{12}, y_{12})\} \) in the plane.
Quadtree Operations

- **Search**: Analogous to binary search trees
- **Insert**:
  - Search for the point
  - Split the leaf if there are two points
- **Delete**:
  - Search for the point
  - Remove the point
  - If its parent has only one child left, delete that child and continue the process toward the root.
Quadtree: Range Search

\[\text{QTree-RangeSearch}(T, R)\]
\(T\): A quadtree node, \(R\): Query rectangle
1. \(\text{if } (T \text{ is a leaf}) \text{ then}\)
2. \(\text{if } (T.point \in R) \text{ then}\)
3. \(\text{report } T.point\)
4. \(\text{for each child } C \text{ of } T \text{ do}\)
5. \(\text{if } C.region \cap R \neq \emptyset \text{ then}\)
6. \(\text{QTree-RangeSearch}(C, R)\)

- **spread factor** of points \(P\): \(\beta(P) = \frac{d_{\text{max}}}{d_{\text{min}}}\)
- \(d_{\text{max}}(d_{\text{min}})\): maximum (minimum) distance between two points in \(P\)
- **height** of quadtree: \(h \in \Theta(\log_2 \frac{d_{\text{max}}}{d_{\text{min}}})\)
- Complexity to build initial tree: \(\Theta(nh)\)
- Complexity of range search: \(\Theta(nh)\) even if the answer is \(\emptyset\)
Quadtree Conclusion

- Very easy to compute and handle
- No complicated arithmetic, only divisions by 2 (usually the boundary box is padded to get a power of two).
- Space wasteful
- Major drawback: can have very large height for certain nonuniform distributions of points
- Easily generates to higher dimensions (octrees, etc.).
kd-trees

- We have \( n \) points \( P = \{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\} \) in the plane.
- Quadtrees split square into quadrants regardless of where points actually lie.
- kd-tree idea: Split the points into two (roughly) equal subsets.
- How to build a kd-tree on \( P \):
  - Split \( P \) into two equal subsets using a vertical line.
  - Split each of the two subsets into two equal pieces using horizontal lines.
  - Continue splitting, alternating vertical and horizontal lines, until every point is in a separate region.
- Complexity: \( \Theta(n \log n) \), height of the tree: \( \Theta(\log n) \).
kd-trees

- We have \( n \) points \( P = \{ (x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1}) \} \) in the plane.
- Quadtrees split square into quadrants regardless of where points actually lie.
- kd-tree idea: Split the points into two (roughly) equal subsets.
- More details:
  - Initially, we sort the \( n \) points according to their \( x \)-coordinates.
  - The root of the tree is the point with median \( x \)-coordinate (index \( \lfloor n/2 \rfloor \) in the sorted list).
  - All other points with \( x \) coordinate less than or equal to this go into the left subtree; points with larger \( x \)-coordinate go in the right subtree.
  - At alternating levels, we sort and split according to \( y \)-coordinates instead.

- Complexity: \( \Theta(n \log n) \), height of the tree: \( \Theta(\log n) \)
kd-trees

- **kd-tree idea**: Split the points into two (roughly) equal subsets
- **A balanced binary tree**

```
    p0        p4
   / \    /  \    /  \
  p3  p9  p8   p5  p6
   \    \     \     \   \  
    p1  p2    p7     p7
```

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kd-trees

- **kd-tree idea**: Split the points into two (roughly) equal subsets
- A *balanced* binary tree
kd-trees

- kd-tree idea: Split the points into two (roughly) equal subsets
- A balanced binary tree
kd-trees

- **kd-tree idea**: Split the points into two (roughly) equal subsets
- A **balanced** binary tree
kd-trees

- kd-tree idea: Split the points into two (roughly) equal subsets
- A **balanced** binary tree
kd-tree: Range Search

kd-rangeSearch(T, R)
T: A kd-tree node, R: Query rectangle
1. if T is empty then return
2. if T.point ∈ R then
3. report T.point
4. for each child C of T do
5. if C.region ∩ R ≠ ∅ then
6. kd-rangeSearch(C, R)
kd-tree: Range Search

\textit{kd-rangeSearch}(T, R, split[← ‘x’])

\begin{itemize}
\item \textbf{T: A} kd-tree node, \textbf{R: Query rectangle}
\item 1. if \text{\textit{T}} is empty then return
\item 2. if \text{T.point} ∈ \text{R} then
\item 3. \textbf{report} \text{T.point}
\item 4. if \text{split} = \text{‘x’} then
\item 5. if \text{T.point}.\text{x} ≥ \text{R.leftSide} then
\item 6. \textit{kd-rangeSearch}(T.left, R, ‘y’)
\item 7. if \text{T.point}.\text{x} < \text{R.rightSide} then
\item 8. \textit{kd-rangeSearch}(T.right, R, ‘y’)
\item 9. if \text{split} = \text{‘y’} then
\item 10. if \text{T.point}.\text{y} ≥ \text{R.bottomSide} then
\item 11. \textit{kd-rangeSearch}(T.left, R, ‘x’)
\item 12. if \text{T.point}.\text{y} < \text{R.topSide} then
\item 13. \textit{kd-rangeSearch}(T.right, R, ‘x’)
\end{itemize}
kd-tree: Range Search Complexity

- The complexity is $O(k + U)$ where $k$ is the number of keys reported and $U$ is the number of regions we go to but unsuccessfully.
- $U$ corresponds to the number of regions which intersect but are not fully in $R$.
- Those regions have to intersect one of the four sides of $R$.
- $Q(n)$: Maximum number of regions in a kd-tree with $n$ points that intersect a vertical (horizontal) line.
- $Q(n)$ satisfies the following recurrence relation:
  \[
  Q(n) = 2Q(n/4) + O(1)
  \]
- It solves to $Q(n) = O(\sqrt{n})$.
- Therefore, the complexity of range search in kd-trees is $O(k + \sqrt{n})$. 
kd-tree: Higher Dimensions

- kd-trees for $d$-dimensional space
  - At the root the point set is partitioned based on the first coordinate
  - At the children of the root the partition is based on the second coordinate
  - At depth $d-1$ the partition is based on the last coordinate
  - At depth $d$ we start all over again, partitioning on first coordinate

- Storage: $O(n)$
- Construction time: $O(n \log n)$
- Range query time: $O(n^{1-1/d} + k)$

(Note: $d$ is considered to be a constant.)
Range Trees

- We have $n$ points $P = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{n-1}, y_{n-1})\}$ in the plane.
- A range tree is a tree of trees (a multi-level data structure).
- How to build a range tree on $P$:
  - Build a balanced binary search tree $\tau$ determined by the $x$-coordinates of the $n$ points.
  - For every node $v \in \tau$, build a balanced binary search tree $\tau_{assoc}(v)$ (associated structure of $\tau$) determined by the $y$-coordinates of the nodes in the subtree of $\tau$ with root node $v$. 
Range Tree Structure

Figure 5.6
A 2-dimensional range tree returns the root of a 2-dimensional range tree $T$ of $P$. As in the previous section, we assume that no two points have the same $x$- or $y$-coordinate. We shall get rid of this assumption in Section 5.5.

Algorithm BUILD2D RANGE TREE($P$)

Input. A set $P$ of points in the plane.
Output. The root of a 2-dimensional range tree.

1. Construct the associated structure: Build a binary search tree $T_{assoc}$ on the set $P_y$ of $y$-coordinates of the points in $P$. Store at the leaves of $T_{assoc}$ not just the $y$-coordinate of the points in $P_y$, but the points themselves.

2. if $P$ contains only one point
3. then Create a leaf $ν$ storing this point, and make $T_{assoc}$ the associated structure of $ν$.
4. else Split $P$ into two subsets; one subset $P_{left}$ contains the points with $x$-coordinate less than or equal to $x_{mid}$, the median $x$-coordinate, and the other subset $P_{right}$ contains the points with $x$-coordinate larger than $x_{mid}$.
5. $ν_{left} ← \text{BUILD2D RANGE TREE}(P_{left})$
6. $ν_{right} ← \text{BUILD2D RANGE TREE}(P_{right})$
7. Create a node $ν$ storing $x_{mid}$, make $ν_{left}$ the left child of $ν$, make $ν_{right}$ the right child of $ν$, and make $T_{assoc}$ the associated structure of $ν$.
8. return $ν$

Note that in the leaves of the associated structures we do not just store the $y$-coordinate of the points but the points themselves. This is important because, when searching the associated structures, we need to report the points and not just the $y$-coordinates.

Lemma 5.6
A range tree on a set of $n$ points in the plane requires $O(n \log n)$ storage.

Proof.
A point $p$ in $P$ is stored only in the associated structure of nodes on the path in $T$ towards the leaf containing $p$. Hence, for all nodes at a given depth of $T$,
Range Trees: Operations

- **Search**: trivially as in a binary search tree
- **Insert**: insert point in $\tau$ by $x$-coordinate
- From inserted leaf, walk back up to the root and insert the point in all associated trees $\tau_{assoc}(v)$ of nodes $v$ on path to the root
- **Delete**: analogous to insertion
- **Note**: re-balancing is a problem!
Range Trees: Range Search

- A two stage process
- To perform a range search query $R = [x_1, x_2] \times [y_1, y_2]$:
  - Perform a range search (on the $x$-coordinates) for the interval $[x_1, x_2]$ in $\tau$ ($\text{BST-RangeSearch}(\tau, x_1, x_2)$)
  - For every outside node, do nothing.
  - For every “top” inside node $v$, perform a range search (on the $y$-coordinates) for the interval $[y_1, y_2]$ in $\tau_{assoc}(v)$. During the range search of $\tau_{assoc}(v)$, do not check any $x$-coordinates (they are all within range).
  - For every boundary node, test to see if the corresponding point is within the region $R$.

- Running time: $O(k + \log^2 n)$
- Range tree space usage: $O(n \log n)$
Range Trees: Higher Dimensions

- Range trees for $d$-dimensional space
  - Storage: $O(n (\log n)^{d-1})$
  - Construction time: $O(n (\log n)^{d-1})$
  - Range query time: $O((\log n)^d + k)$

(Note: $d$ is considered to be a constant.)
Range Trees: Higher Dimensions

- Space/time trade-off
  - Storage: $O(n \log^n d^{-1})$
  - Construction time: $O(n \log^n d^{-1})$
  - Range query time: $O((\log^n d + k)$

(ktrees: $O(n)$

(ktrees: $O(n \log n)$

(ktrees: $O(n^{1-1/d} + k)$

(Note: $d$ is considered to be a constant.)