Module 9: Tries and String Matching

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Pattern Matching

- Search for a string (pattern) in a large body of text
- $T[0..n-1]$ – The text (or haystack) being searched within
- $P[0..m-1]$ – The pattern (or needle) being searched for
- Strings over alphabet $\Sigma$
- Return the first $i$ such that

$$P[j] = T[i+j] \quad \text{for} \quad 0 \leq j \leq m-1$$

- This is the first occurrence of $P$ in $T$
- If $P$ does not occur in $T$, return FAIL

Applications:
  - Information Retrieval (text editors, search engines)
  - Bioinformatics
  - Data Mining
Pattern Matching

Example:
- $T = \text{“Where is he?”}$
- $P_1 = \text{“he”}$
- $P_2 = \text{“who”}$

Definitions:
- **Substring** $T[i..j]$ with $0 \leq i \leq j < n$: a string of length $j - i + 1$ which consists of characters $T[i], \ldots, T[j]$ in order
- A **prefix** of $T$: a substring $T[0..i]$ of $T$ for some $0 \leq i < n$
- A **suffix** of $T$: a substring $T[i..n - 1]$ of $T$ for some $0 \leq i \leq n - 1$

General Idea of Algorithms

Pattern matching algorithms consist of **guesses** and **checks**:
- A **guess** is a position $i$ such that $P$ might start at $T[i]$. Valid guesses (initially) are $0 \leq i \leq n - m$.
- A **check** of a guess is a single position $j$ with $0 \leq j < m$ where we compare $T[i + j]$ to $P[j]$. We must perform $m$ checks of a single **correct** guess, but may make (many) fewer checks of an **incorrect** guess.

We will diagram a single run of any pattern matching algorithm by a matrix of checks, where each row represents a single guess.
Brute-force Algorithm

Idea: Check every possible guess.

BruteforcePM(T[0..n − 1], P[0..m − 1])
T: String of length n (text), P: String of length m (pattern)
1. for i ← 0 to n − m do
2. match ← true
3. j ← 0
4. while j < m and match do
5. if T[i + j] = P[j] then
6. j ← j + 1
7. else
8. match ← false
9. if match then
10. return i
11. return FAIL

Example

Example: T = abbbababbab, P = abba

What is the worst possible input?
P = am−1b, T = an

Worst case performance Θ((n − m + 1)m)
m ≤ n/2 ⇒ Θ(mn)
Pattern Matching

More sophisticated algorithms

- Deterministic finite automata (DFA)
- KMP, Boyer-Moore and Rabin-Karp
- Do extra preprocessing on the pattern $P$
- We eliminate guesses based on completed matches and mismatches.

String matching with finite automata

There is a string-matching automaton for every pattern $P$. It is constructed from the pattern in a preprocessing step before it can be used to search the text string.

**Example:** Automaton for the pattern $P = \text{ababaca}$
String matching with finite automata

Let $P$ be the pattern to search, of length $m$. Then
- the states of the automaton are $0, \ldots, m$
- the transition function $\delta$ of the automaton is defined as follows, for a state $q$ and a character $c$ in $\Sigma$:
  \[
  \delta(q, c) = \ell(P[0..q-1]c),
  \]
  where
- $P[0..q-1]c$ is the concatenation of $P[0..q-1]$ and $c$
- for a string $s$, $\ell(s) \in \{0, \ldots, m\}$ is the length of the longest prefix of $P$ that is also a suffix of $s$.

Graphically, this corresponds to

Idea of proof: the state after reading $T[i]$ is $\ell(T[0..i])$. 
String matching with finite automata

- Matching time on a text string of length $n$ is $\Theta(n)$

- This does not include the preprocessing time required to compute the transition function $\delta$. There exists an algorithm with $O(m|\Sigma|)$ preprocessing time.

- Altogether, we can find all occurrences of a length-$m$ pattern in a length-$n$ text over a finite alphabet $\Sigma$ with $O(m|\Sigma|)$ preprocessing time and $\Theta(n)$ matching time.

KMP Algorithm

- Knuth-Morris-Pratt algorithm (1977)
- Compares the pattern to the text in left-to-right
- Shifts the pattern more intelligently than the brute-force algorithm
- When a mismatch occurs, how much can we shift the pattern (reusing knowledge from previous matches)?

$$T = \begin{array}{cccccccc}
\text{a} & \text{b} & \text{c} & \text{d} & \text{c} & \text{a} & \text{b} & \text{c} \\
\text{a} & \text{b} & \text{c} & \text{d} & \text{c} & \text{a} & \text{b} & \text{a} \\
\text{a} & \text{b} & \text{c} & \text{d} & \text{c} & \text{a} & \text{b} & \text{c}
\end{array}$$

- **KMP Answer**: this depends on the largest prefix of $P[0..j]$ that is a suffix of $P[1..j]$
KMP Failure Array

Suppose we have a match up to position $T[i - 1] = P[j - 1]$, but not at the next position.

Define $F[j - 1]$ as the index we will have to check in $P$, after we bring the pattern to its next possible position (previous example: $j = 6$, $F[5] = 2$).

This can be computed by trying all sliding positions until finding the first one matching the text (as in previous example). We can do better:

- any possible sliding position corresponds to a prefix of $P[0..j - 1]$ that is also a *strict* suffix of it = a suffix of $P[1..j - 1]$
- the next possible sliding position corresponds to the **largest** such prefix / suffix
- we let $F[j - 1]$ be the length of this prefix / suffix.
KMP Failure Array

Schematically:

\[ T \]

\[ P \]

\[ j \]

\[ j - 1 \]

\[ F[j - 1] \]

\[ F[j - 1] \]

no need to check
for matching with \( T \)
comparing with \( T \) starts from here

\[ T \]

\[ P \]

\[ j \]

\[ F[j - 1] \]

\[ F[j - 1] \]

slide

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KMP Failure Array

- \( F[0] = 0 \)
- \( F[j] \), for \( j > 0 \), is the length of the largest prefix of \( P[0..j] \) that is also a suffix of \( P[1..j] \)
- Consider \( P = \text{abacaba} \)

<table>
<thead>
<tr>
<th>( j )</th>
<th>( P[1..j] )</th>
<th>( P )</th>
<th>( F[j] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>—</td>
<td>\text{abacaba}</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>b</td>
<td>\text{abacaba}</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>ba</td>
<td>\text{abacaba}</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>bac</td>
<td>\text{abacaba}</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>baca</td>
<td>\text{abacaba}</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>bacab</td>
<td>\text{abacaba}</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>bacaba</td>
<td>\text{abacaba}</td>
<td>3</td>
</tr>
</tbody>
</table>

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Computing the Failure Array

\[\text{failureArray}(P)\]

\(P\): String of length \(m\) (pattern)

1. \(F[0] \leftarrow 0\)
2. \(i \leftarrow 1\)
3. \(j \leftarrow 0\)
4. \textbf{while} \(i < m\) \textbf{do}
5. \quad \textbf{if} \(P[i] = P[j]\) \textbf{then}
6. \quad \quad \(F[i] \leftarrow j + 1\)
7. \quad \quad \(i \leftarrow i + 1\)
8. \quad \quad \(j \leftarrow j + 1\)
9. \quad \textbf{else if} \(j > 0\) \textbf{then}
10. \quad \quad \(j \leftarrow F[j - 1]\)
11. \quad \textbf{else}
12. \quad \quad \(F[i] \leftarrow 0\)
13. \quad \(i \leftarrow i + 1\)

KMP Algorithm

\[\text{KMP}(T, P), \text{to return the first match}\]

\(T\): String of length \(n\) (text), \(P\): String of length \(m\) (pattern)

1. \(F \leftarrow \text{failureArray}(P)\)
2. \(i \leftarrow 0\)
3. \(j \leftarrow 0\)
4. \textbf{while} \(i < n\) \textbf{do}
5. \quad \textbf{if} \(T[i] = P[j]\) \textbf{then}
6. \quad \quad \textbf{if} \(j = m - 1\) \textbf{then}
7. \quad \quad \quad \textbf{return} \(i - j \// \text{match}\)
8. \quad \quad \textbf{else}
9. \quad \quad \quad \(i \leftarrow i + 1\)
10. \quad \quad \quad \(j \leftarrow j + 1\)
11. \quad \textbf{else}
12. \quad \quad \textbf{if} \(j > 0\) \textbf{then}
13. \quad \quad \quad \(j \leftarrow F[j - 1]\)
14. \quad \quad \textbf{else}
15. \quad \quad \quad \(i \leftarrow i + 1\)
16. \quad \textbf{return} \(-1 \// \text{no match}\)
KMP: Example

\[ P = \text{abacaba} \]

\[ T = \text{abaxyabacabbaababacaba} \]

<table>
<thead>
<tr>
<th>( j )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F[j] )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Exercise: continue with \( T = \text{abaxyabacabbaababacaba} \)

KMP: Analysis

failureArray

- At each iteration of the while loop, at least one of the following happens:
  1. \( i \) increases by one, or
  2. the index \( i - j \) increases by at least one \((F[j - 1] < j)\)
- There are no more than \( 2m \) iterations of the while loop
- Running time: \( \Theta(m) \)

KMP

- failureArray can be computed in \( \Theta(m) \) time
- At each iteration of the while loop, at least one of the following happens:
  1. \( i \) increases by one, or
  2. the index \( i - j \) increases by at least one \((F[j - 1] < j)\)
- There are no more than \( 2n \) iterations of the while loop
- Running time: \( \Theta(n) \)
Boyer-Moore Algorithm

Based on three key ideas:

- **Reverse-order searching**: Compare \( P \) with a subsequence of \( T \) moving backwards.

- **Bad character jumps**: When a mismatch occurs at \( T[i] = c \)
  - If \( P \) contains \( c \), we can shift \( P \) to align the last occurrence of \( c \) in \( P \) with \( T[i] \)
  - Otherwise, we can shift \( P \) to align \( P[0] \) with \( T[i + 1] \)

- **Good suffix jumps**: If we have already matched a suffix of \( P \), then get a mismatch, we can shift \( P \) forward to align with the previous occurrence of that suffix (with a mismatch from the suffix we read). If none exists, look for the longest prefix of \( P \) that is a suffix of what we read. Similar to failure array in KMP.

- Can skip large parts of \( T \)

Bad character examples

\[
P = a \quad l \quad d \quad o \\
T = w \quad h \quad e \quad r \quad e \quad i \quad s \quad w \quad a \quad l \quad d \quad o
\]

- \( o \) \( o \) \( a \) \( l \) \( d \) \( o \)

6 comparisons (checks)

\[
P = m \quad o \quad o \quad r \quad e \\
T = b \quad o \quad y \quad e \quad r \quad m \quad o \quad o \quad r \quad e
\]

- \( e \) \( (r) \) \( e \) \( (m) \) \( o \) \( o \) \( r \) \( e \)

7 comparisons (checks)
Good suffix examples

\[ P = \text{sells shells} \]

\[
\begin{array}{ccccccc}
\text{s h e i l a} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{s e l l s} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{s h e l l s} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{s h e l l s} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{s (e)} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{s h e l l s} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\end{array}
\]

\[ P = \text{odetofood} \]

\[
\begin{array}{ccccccc}
\text{i l i k e f o o d f r o m m e x i c o} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{o f f o o d} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{(e)} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{(o)} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{d} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{d} & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\end{array}
\]

- Good suffix moves further than bad character for 2nd guess.
- Bad character moves further than good suffix for 3rd guess.
- This is out of range, so \text{pattern not found}.

Last-Occurrence Function

- **Preprocess** the pattern \(P\) and the alphabet \(\Sigma\)
- Build the last-occurrence function \(L\) mapping \(\Sigma\) to integers
- \(L(c)\) is defined as
  - the largest index \(i\) such that \(P[i] = c\) or
  - \(-1\) if no such index exists
- Example: \(\Sigma = \{a, b, c, d\}\), \(P = abacab\)

\[
\begin{array}{|c|c|c|c|c|}
\hline
c & a & b & c & d \\
\hline
L(c) & 4 & 5 & 3 & -1 \\
\hline
\end{array}
\]

- The last-occurrence function can be computed in time \(O(m + |\Sigma|)\)
- In practice, \(L\) is stored in a size-\(|\Sigma|\) array.
Good Suffix array

- Again, we **preprocess** $P$ to build a table.
- **Suffix skip array** $S$ of size $m$: for $0 \leq i < m$, $S[i]$ is the largest index $j$ such that $P[i+1..m-1] = P[j+1..j+m-1-i]$ and $P[j] \neq P[i]$.
- **Note:** in this calculation, any negative indices are considered to make the given condition true (these correspond to letters that we might not have checked yet).
- Similar to KMP failure array, with an extra condition.
- Computed similarly to KMP failure array in $\Theta(m)$ time.

**Example:** $P = \text{bonobobo}$

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[i]$</td>
<td>b</td>
<td>o</td>
<td>n</td>
<td>o</td>
<td>b</td>
<td>o</td>
<td>b</td>
<td>o</td>
</tr>
<tr>
<td>$S[i]$</td>
<td>−6</td>
<td>−5</td>
<td>−4</td>
<td>−3</td>
<td>2</td>
<td>−1</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

- Computed similarly to KMP failure array in $\Theta(m)$ time.
Boyer-Moore Algorithm

\begin{verbatim}
boyer-moore(T,P)
1.     \( L \leftarrow \) last occurrence array computed from \( P \)
2.     \( S \leftarrow \) good suffix array computed from \( P \)
3.     \( i \leftarrow m-1, \quad j \leftarrow m-1 \)
4.     while \( i < n \) and \( j \geq 0 \) do
5.         if \( T[i] = P[j] \) then
6.             \( i \leftarrow i - 1 \)
7.             \( j \leftarrow j - 1 \)
8.         else
9.             \( i \leftarrow i + m - 1 - \min(L[T[i]], S[j]) \)
10.            \( j \leftarrow m - 1 \)
11.        if \( j = -1 \) return \( i + 1 \)
12.    else return FAIL
\end{verbatim}

**Exercise:** Prove that \( i - j \) always increases on lines 9–10.

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Boyer-Moore algorithm conclusion

- Worst-case running time \( \in O(n + |\Sigma|) \)
- This complexity is difficult to prove.
- Worst-case running time \( O(nm) \) if we want to report all occurrences
- On typical **English text** the algorithm probes approximately 25\% of the characters in \( T \)
- Faster than KMP in practice on English text.
Rabin-Karp Fingerprint Algorithm

**Idea:** use hashing
- Compute hash function for each text position
- No explicit hash table: just compare with pattern hash
- If a match of the hash value of the pattern and a text position found, then compares the pattern with the substring by naive approach

**Example:**
Hash "table" size = 97
Search Pattern $P$: 5 9 2 6 5
Search Text $T$: 3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6
Hash function: $h(x) = x \mod 97$ and $h(P) = 95$.
31415 mod 97 = 84
14159 mod 97 = 94
41592 mod 97 = 76
15926 mod 97 = 18
59265 mod 97 = 95

Guaranteeing correctness
Need full compare on hash match to guard against collisions
- 59265 mod 97 = 95
- 59362 mod 97 = 95

Running time
- Hash function depends on $m$ characters
- Running time is $\Theta(mn)$ for search miss (how can we fix this?)
Rabin-Karp Fingerprint Algorithm

The initial hashes are called **fingerprints**. Rabin & Karp discovered a way to update these fingerprints in constant time.

**Idea:**
To go from the hash of a substring in the text string to the next hash value only requires constant time.

- Use previous hash to compute next hash
- \(O(1)\) time per hash, except first one

**Example:**
- Pre-compute: \(10000 \mod 97 = 9\)
- Previous hash: \(41592 \mod 97 = 76\)
- Next hash: \(15926 \mod 97 = ??\)

**Observation:**
\[
15926 \mod 97 = (41592 - (4 \times 10000)) \times 10 + 6 \\
= (76 - (4 \times 9)) \times 10 + 6 \\
= 406 \\
= 18
\]
Rabin-Karp Fingerprint Algorithm

- Choose table size at random to be huge prime
- Expected running time is $O(m + n)$
- $\Theta(mn)$ worst-case, but this is (unbelievably) unlikely

Main advantage:
- Extends to 2d patterns and other generalizations

Suffix Tries and Suffix Trees

- What if we want to search for many patterns $P$ within the same fixed text $T$?
- Idea: Preprocess the text $T$ rather than the pattern $P$
- Observation: $P$ is a substring of $T$ if and only if $P$ is a prefix of some suffix of $T$.

We will call a trie that stores all suffixes of a text $T$ a suffix trie, and the compressed suffix trie of $T$ a suffix tree.
Suffix Trees

- Build the suffix trie, i.e. the trie containing all the suffixes of the text
- Build the suffix tree by compressing the trie above (like in Patricia trees)
- Store two indexes $l, r$ on each node $v$ (both internal nodes and leaves) where node $v$ corresponds to substring $T[l..r]$
Suffix Trees: Pattern Matching

To search for pattern $P$ of length $m$:

- Similar to Search in compressed trie with the difference that we are looking for a prefix match rather than a complete match.
- If we reach a leaf with a corresponding string length less than $m$, then search is unsuccessful.
- Otherwise, we reach a node $v$ (leaf or internal) with a corresponding string length of at least $m$.
- It only suffices to check the first $m$ characters against the substring of the text between indices of the node, to see if there indeed is a match.
- We can then visit all children of the node to report all matches.
Suffix Tree: Example

$T = \text{bananaban}$

$P = \text{ana}$

- Suffix Tree Diagram

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T[i]$</td>
<td>b</td>
<td>a</td>
<td>n</td>
<td>a</td>
<td>n</td>
<td>b</td>
<td>a</td>
<td>n</td>
<td>$$</td>
<td></td>
</tr>
</tbody>
</table>

Suffix Tree: Example

$T = \text{bananaban}$

$P = \text{ban}$

- Suffix Tree Diagram

<table>
<thead>
<tr>
<th>$i$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T[i]$</td>
<td>b</td>
<td>a</td>
<td>n</td>
<td>a</td>
<td>n</td>
<td>b</td>
<td>a</td>
<td>n</td>
<td>$$</td>
<td></td>
</tr>
</tbody>
</table>
Suffix Tree: Example
T = bananaban
P = nana

Suffix Tree: Example
T = bananaban
P = bbn not found
### Pattern Matching Conclusion

<table>
<thead>
<tr>
<th></th>
<th>Brute-Force</th>
<th>DFA</th>
<th>KMP</th>
<th>BM</th>
<th>RK</th>
<th>Suffix trees</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preproc.</strong></td>
<td>–</td>
<td>$O(m</td>
<td>Σ</td>
<td>)$</td>
<td>$O(m)$</td>
<td>$O(m +</td>
</tr>
<tr>
<td><strong>Search time:</strong></td>
<td>$O(nm)$</td>
<td>$O(n)$</td>
<td>$O(n)$</td>
<td>$O(n)$ (often better)</td>
<td>$O(n + m)$ (expected)</td>
<td>$O(m)$</td>
</tr>
<tr>
<td><strong>Extra space:</strong></td>
<td>–</td>
<td>$O(m</td>
<td>Σ</td>
<td>)$</td>
<td>$O(m)$</td>
<td>$O(m +</td>
</tr>
</tbody>
</table>

Haque, Irvine, Smith (SCS, UW)  
CS240 - Module 9  
Spring 2017  
43 / 43