Module 9: Tries and String Matching

CS 240 - Data Structures and Data Management

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Based on lecture notes by many previous cs240 instructors

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Pattern Matching

- Search for a string (pattern) in a large body of text
- \( T[0..n-1] \) – The text (or haystack) being searched within
- \( P[0..m-1] \) – The pattern (or needle) being searched for
- Strings over alphabet \( \Sigma \)
- Return the first \( i \) such that
  \[
  P[j] = T[i+j] \quad \text{for} \quad 0 \leq j \leq m-1
  \]
- This is the first occurrence of \( P \) in \( T \)
- If \( P \) does not occur in \( T \), return FAIL
- Applications:
  - Information Retrieval (text editors, search engines)
  - Bioinformatics
  - Data Mining

Example:
- \( T = \) “Where is he?”
- \( P_1 = \) “he”
- \( P_2 = \) “who”

Definitions:
- Substring \( T[i..j] \) \( 0 \leq i \leq j < n \): a string of length \( j - i + 1 \) which consists of characters \( T[i], \ldots, T[j] \) in order
- A prefix of \( T \): a substring \( T[0..i] \) of \( T \) for some \( 0 \leq i < n \)
- A suffix of \( T \): a substring \( T[i..n-1] \) of \( T \) for some \( 0 \leq i \leq n - 1 \)
General Idea of Algorithms

Pattern matching algorithms consist of guesses and checks:
- A **guess** is a position \( i \) such that \( P \) might start at \( T[i] \).
  Valid guesses (initially) are \( 0 \leq i \leq n - m \).
- A **check** of a guess is a single position \( j \) with \( 0 < j < m \) where we compare \( T[i + j] \) to \( P[j] \). We must perform \( m \) checks of a single correct guess, but may make (many) fewer checks of an incorrect guess.

We will diagram a single run of any pattern matching algorithm by a matrix of checks, where each row represents a single guess.

**Brute-force Algorithm**

**Idea:** Check every possible guess.

\[
\text{BruteforcePM}(T[0..n-1], P[0..m-1])
\]

- \( T \): String of length \( n \) (text), \( P \): String of length \( m \) (pattern)
- 1. for \( i \) ← 0 to \( n - m \) do
- 2. \( \text{match} \) ← true
- 3. \( j \) ← 0
- 4. while \( j < m \) and \( \text{match} \) do
- 5. if \( T[i + j] = P[j] \) then
- 6. \( j \) ← \( j + 1 \)
- 7. else
- 8. \( \text{match} \) ← false
- 9. if \( \text{match} \) then
- 10. return \( i \)
- 11. return \( \text{FAIL} \)

**Example**

- Example: \( T = \text{abbbabbbab}, P = \text{abba} \)

- What is the worst possible input?
  - \( P = a^{m-1}b \), \( T = a^n \)
  - Worst case performance \( \Theta((n - m + 1)m) \)
  - \( m \leq n/2 \) ⇒ \( \Theta(mn) \)
Pattern Matching

More sophisticated algorithms
- Deterministic finite automata (DFA)
- KMP, Boyer-Moore and Rabin-Karp
- Do extra preprocessing on the pattern $P$
- We eliminate guesses based on completed matches and mismatches.

String matching with finite automata
There is a string-matching automaton for every pattern $P$. It is constructed from the pattern in a preprocessing step before it can be used to search the text string.

**Example:** Automaton for the pattern $P = ababaca$

![String matching automaton diagram](image)

String matching with finite automata
Let $P$ the pattern to search, of length $m$. Then
- the states of the automaton are $0, \ldots, m$
- the transition function $\delta$ of the automaton is defined as follows, for a state $q$ and a character $c$ in $\Sigma$:

$$\delta(q, c) = \ell(P[0..q-1]c),$$

where
- $P[0..q-1]c$ is the concatenation of $P[0..q-1]$ and $c$
- for a string $s$, $\ell(s) \in \{0, \ldots, m\}$ is the length of the longest prefix of $P$ that is also a suffix of $s$.

Graphically, this corresponds to

![Graphical representation](image)
String matching with finite automata

Let $T$ be the text string of length $n$, $P$ the pattern to search of length $m$ and $\delta$ the transition function of a finite automaton for pattern $P$.

**FINITE-AUTOMATON-MATCHER($T, \delta, m$)**

$n \leftarrow \text{length}[T]$
$q \leftarrow 0$
for $i \leftarrow 0$ to $n - 1$ do
$q \leftarrow \delta(q, T[i])$
if $q = m$
then print "Pattern occurs with shift" $i - (m - 1)$

**Idea of proof:** the state after reading $T[i]$ is $\ell(T[0..i])$.

Matching time on a text string of length $n$ is $\Theta(n)$

This does not include the preprocessing time required to compute the transition function $\delta$. There exists an algorithm with $O(m|\Sigma|)$ preprocessing time.

Altogether, we can find all occurrences of a length-$m$ pattern in a length-$n$ text over a finite alphabet $\Sigma$ with $O(m|\Sigma|)$ preprocessing time and $\Theta(n)$ matching time.

KMP Algorithm

- Knuth-Morris-Pratt algorithm (1977)
- Compares the pattern to the text in left-to-right
- Shifts the pattern more intelligently than the brute-force algorithm
- When a mismatch occurs, how much can we shift the pattern (reusing knowledge from previous matches)?

$$T = \begin{array}{cccccccc}
  a & b & c & d & c & a & b & c & ? & ? \\
  a & b & c & d & c & a & b & a & a & a \\
  a & b & c & d & c & a & a & a & a & a
\end{array}$$

- **KMP Answer:** this depends on the largest prefix of $P[0..j]$ that is a suffix of $P[1..j]$
KMP Failure Array

Suppose we have a match up to position $T[i - 1] = P[j - 1]$, but not at the next position.

Define $F[j - 1]$ as the index we will have to check in $P$, after we bring the pattern to its next possible position (previous example: $j = 6$, $F[5] = 2$).

This can be computed by trying all sliding positions until finding the first one matching the text (as in previous example). We can do better:

- any possible sliding position corresponds to a prefix of $P[0..j - 1]$ that is also a strict suffix of it = a suffix of $P[1..j - 1]$
- the next possible sliding position corresponds to the largest such prefix / suffix
- we let $F[j - 1]$ be the length of this prefix / suffix.

KMP Failure Array

Schematically:

- $T$: a b b c a b c d...
- $P$: a b b c a b a a

what next slide would match with the text?
KMP Failure Array

- $F[0] = 0$
- $F[j]$, for $j > 0$, is the length of the largest prefix of $P[0..j]$ that is also a suffix of $P[1..j]$
- Consider $P = \text{abacaba}$

<table>
<thead>
<tr>
<th>$j$</th>
<th>$P[1..j]$</th>
<th>$P$</th>
<th>$F[j]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>—</td>
<td>a</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>b</td>
<td>abacaba</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>ba</td>
<td>abacaba</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>bac</td>
<td>abacaba</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>baca</td>
<td>abacaba 1</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>bacab</td>
<td>abacaba 2</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>bacaba</td>
<td>abacaba 3</td>
<td></td>
</tr>
</tbody>
</table>

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Computing the Failure Array

```
def failureArray(P):
P: String of length m (pattern)
1. F[0] ← 0
2. i ← 1
3. j ← 0
4. while i < m do
5.   if P[i] = P[j] then
6.     F[i] ← j + 1
7.     i ← i + 1
8.     j ← j + 1
9. else if j > 0 then
10.    j ← F[j - 1]
11. else
12.    F[i] ← 0
13.    i ← i + 1
```

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KMP Algorithm

```
def KMP(T, P):
T: String of length n (text), P: String of length m (pattern)
1. F ← failureArray(P)
2. i ← 0
3. j ← 0
4. while i < n do
5.   if T[i] = P[j] then
6.     if j = m - 1 then
7.       return i - j //match
8.     else
9.       i ← i + 1
10.      j ← j + 1
11. else
12.   if j > 0 then
13.     j ← F[j - 1]
14. else
15.     i ← i + 1
16. return -1 // no match
```

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KMP: Example

\[ P = \text{abacaba} \]
\[ T = \text{abaxyabacabbaabacaba} \]

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 \\
a & b & a & c & y & a & b \\
\hline
\end{array}
\]

Exercise: continue with \( T = \text{abaxyabacabbaabacaba} \)

KMP: Analysis

failureArray

- At each iteration of the while loop, at least one of the following happens:
  - \( i \) increases by one, or
  - the index \( i - j \) increases by at least one \((F[j - 1] < j)\)
- There are no more than 2\(m\) iterations of the while loop
- Running time: \( \Theta(m) \)

KMP

- failureArray can be computed in \( \Theta(m) \) time
- At each iteration of the while loop, at least one of the following happens:
  - \( i \) increases by one, or
  - the index \( i - j \) increases by at least one \((F[j - 1] < j)\)
- There are no more than 2\(n\) iterations of the while loop
- Running time: \( \Theta(n) \)

Boyer-Moore Algorithm

Based on three key ideas:

- **Reverse-order searching**: Compare \( P \) with a subsequence of \( T \) moving backwards
- **Bad character jumps**: When a mismatch occurs at \( T[i] = c \)
  - If \( P \) contains \( c \), we can shift \( P \) to align the last occurrence of \( c \) in \( P \) with \( T[i] \)
  - Otherwise, we can shift \( P \) to align \( P[0] \) with \( T[i+1] \)
- **Good suffix jumps**: If we have already matched a suffix of \( P \), then get a mismatch, we can shift \( P \) forward to align with the previous occurrence of that suffix (with a mismatch from the suffix we read). If none exists, look for the longest prefix of \( P \) that is a suffix of what we read. Similar to failure array in KMP.
- Can skip large parts of \( T \)
Bad character examples

\[ P = a l d o \]
\[ T = w h e r e i s w a l d o \]

6 comparisons (checks)

\[ P = m o o r e \]
\[ T = b o y e r m o o r e \]

7 comparisons (checks)

Good suffix examples

\[ P = sells \_ shells \]
\[ s h e i l a \_ s e l l s \_ s h e l l s \]

\[ P = o d e t o f o o d \]
\[ i l i k e f o o d f r o m m e x i c o \]

- Good suffix moves further than bad character for 2nd guess.
- Bad character moves further than good suffix for 3rd guess.
- This is out of range, so pattern not found.

Last-Occurrence Function

- **Preprocess** the pattern \( P \) and the alphabet \( \Sigma \)
- Build the **last-occurrence function** \( L \) mapping \( \Sigma \) to integers
- \( L(c) \) is defined as
  - the largest index \( i \) such that \( P[i] = c \) or
  - \(-1\) if no such index exists
- Example: \( \Sigma = \{a, b, c, d\} \), \( P = abacab \)

<table>
<thead>
<tr>
<th>c</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L(c) )</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>-1</td>
</tr>
</tbody>
</table>

- The last-occurrence function can be computed in time \( O(m + |\Sigma|) \)
- In practice, \( L \) is stored in a size-\( |\Sigma| \) array.
Good Suffix array

- Again, we preprocess $P$ to build a table.
- **Suffix skip array** $S$ of size $m$: for $0 \leq i < m$, $S[i]$ is the largest index $j$ such that $P[i+1..m-1] = P[j+1..m-1-i]$ and $P[j] \neq P[i]$. 
  - **Note**: in this calculation, any negative indices are considered to make the given condition true (these correspond to letters that we might not have checked yet).
- Similar to KMP failure array, with an extra condition.
- Computed similarly to KMP failure array in $\Theta(m)$ time.

Example: $P = \text{bonobobo}$

```
<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P[i]$</td>
<td>b</td>
<td>o</td>
<td>n</td>
<td>o</td>
<td>b</td>
<td>o</td>
<td>b</td>
<td>o</td>
</tr>
<tr>
<td>$S[i]$</td>
<td>-6</td>
<td>-5</td>
<td>-4</td>
<td>-3</td>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>
```

Computed similarly to KMP failure array in $\Theta(m)$ time.

Boyer-Moore Algorithm

```
boyer-moore(T,P)
1. $L \leftarrow$ last occurrence array computed from $P$
2. $S \leftarrow$ good suffix array computed from $P$
3. $i \leftarrow m - 1$, $j \leftarrow m - 1$
4. while $i < n$ and $j \geq 0$ do
5.   if $T[i] = P[j]$ then
6.     $i \leftarrow i - 1$
7.   else
8.     $i \leftarrow i + m - 1 - \min(L[T[i]], S[j])$
9.     $j \leftarrow m - 1$
10. if $j = -1$ return $i + 1$
11. else return FAIL
```

**Exercise**: Prove that $i - j$ always increases on lines 9–10.
Boyer-Moore algorithm conclusion

- Worst-case running time $\in O(n + |\Sigma|)$
- This complexity is difficult to prove.
- Worst-case running time $O(nm)$ if we want to report all occurrences
- On typical English text the algorithm probes approximately 25% of the characters in $T$
- Faster than KMP in practice on English text.

Rabin-Karp Fingerprint Algorithm

**Idea:** use hashing

- Compute hash function for each text position
- No explicit hash table: just compare with pattern hash
- If a match of the hash value of the pattern and a text position found, then compares the pattern with the substring by naive approach

**Example:**
Hash "table" size = 97
Search Pattern $P$: 5 9 2 6 5
Search Text $T$: 3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6
Hash function: $h(x) = x \mod 97$ and $h(P) = 95$.
31415 mod 97 = 84
14159 mod 97 = 94
41592 mod 97 = 76
15926 mod 97 = 18
59265 mod 97 = 95

Guaranteeing correctness

- Need full compare on hash match to guard against collisions
  - $59265 \mod 97 = 95$
  - $59362 \mod 97 = 95$

Running time

- Hash function depends on $m$ characters
- Running time is $\Theta(mn)$ for search miss (how can we fix this?)
Rabin-Karp Fingerprint Algorithm

The initial hashes are called **fingerprints**. Rabin & Karp discovered a way to update these fingerprints in constant time.

**Idea:**
To go from the hash of a substring in the text string to the next hash value only requires constant time.

- Use previous hash to compute next hash
- $O(1)$ time per hash, except first one

---

**Example:**
- Pre-compute: $10000 \mod 97 = 9$
- Previous hash: $41592 \mod 97 = 76$
- Next hash: $15926 \mod 97 = ??$

**Observation:**

\[
15926 \mod 97 = (41592 - (4 \times 10000)) \times 10 + 6
= (76 - (4 \times 9)) \times 10 + 6
= 406
= 18
\]

---

**Rabin-Karp Fingerprint Algorithm**

- Choose table size at random to be huge prime
- Expected running time is $O(m + n)$
- $\Theta(mn)$ worst-case, but this is (unbelievably) unlikely

**Main advantage:**
- Extends to 2d patterns and other generalizations
Suffix Tries and Suffix Trees

- What if we want to search for many patterns $P$ within the same fixed text $T$?
- Idea: Preprocess the text $T$ rather than the pattern $P$
- Observation: $P$ is a substring of $T$ if and only if $P$ is a prefix of some suffix of $T$.

We will call a trie that stores all suffixes of a text $T$ a suffix trie, and the compressed suffix trie of $T$ a suffix tree.

Suffix Trees

- Build the suffix trie, i.e. the trie containing all the suffixes of the text
- Build the suffix tree by compressing the trie above (like in Patricia trees)
- Store two indexes $l, r$ on each node $v$ (both internal nodes and leaves) where node $v$ corresponds to substring $T[l..r]$

Suffix Trie: Example

$T =$bananaban
{bananaban, ananaban, nanaban, anaban, naban, aban, ban, an, n}
Suffix Trees: Pattern Matching

To search for pattern $P$ of length $m$:
- Similar to Search in compressed trie with the difference that we are looking for a prefix match rather than a complete match
- If we reach a leaf with a corresponding string length less than $m$, then search is unsuccessful
- Otherwise, we reach a node $v$ (leaf or internal) with a corresponding string length of at least $m$
- It only suffices to check the first $m$ characters against the substring of the text between indices of the node, to see if there indeed is a match
- We can then visit all children of the node to report all matches

Suffix Tree: Example

$T = \text{bananaban}$
$P = \text{ana}$
Suffix Tree: Example

$T = \text{bananaban}$
$P = \text{ban}$
## Pattern Matching Conclusion

<table>
<thead>
<tr>
<th></th>
<th>Brute-Force</th>
<th>DFA</th>
<th>KMP</th>
<th>BM</th>
<th>RK</th>
<th>Suffix trees</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preproc.</strong></td>
<td>–</td>
<td>(O(m</td>
<td>\Sigma</td>
<td>))</td>
<td>(O(m))</td>
<td>(O(m +</td>
</tr>
<tr>
<td><strong>Search time:</strong></td>
<td>(O(nm))</td>
<td>(O(n))</td>
<td>(O(n)) (often better)</td>
<td>(O(n + m)) (expected)</td>
<td>(O(n))</td>
<td>(O(m))</td>
</tr>
<tr>
<td><strong>Extra space:</strong></td>
<td>–</td>
<td>(O(m</td>
<td>\Sigma</td>
<td>))</td>
<td>(O(m))</td>
<td>(O(m +</td>
</tr>
</tbody>
</table>