# CS 240 – Data Structures and Data Management

## Module 1: Introduction and Asymptotic Analysis

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References: Goodrich & Tamassia 1.1, 1.2, 1.3
Sedgewick 8.2, 8.3

# Outline

# Outline

# Course Objectives: What is this course about?

- When first learning to program, we emphasize *correctness*: does your program output the expected results?
- Starting with this course, we will also be concerned with *efficiency*: is your program using the computer's resources (typically processor time) efficiently?
- We will study efficient methods of *storing*, *accessing*, and performing *operations* on large collections of data.
- Typical operations include: *inserting* new data items, *deleting* data items, *searching* for specific data items, *sorting*.
- **Motivating examples:** Digital Music Collection, English Dictionary

# Course Objectives: What is this course about?

- We will consider various **abstract data types** (ADTs) and how to implement them efficiently using appropriate **data structures**.
- There is a strong emphasis on mathematical analysis in the course.
- Algorithms are presented using pseudocode and analyzed using order notation (big-Oh, etc.).

# Course Topics

- big-Oh analysis
- priority queues and heaps
- sorting, selection
- binary search trees, AVL trees, B-trees
- skip lists
- hashing
- quadtrees, kd-trees
- range search
- tries
- string matching
- data compression

# CS Background

Topics covered in previous courses with relevant sections in [Sedgewick]:

- arrays, linked lists (Sec. 3.2–3.4)
- strings (Sec. 3.6)
- stacks, queues (Sec. 4.2–4.6)
- abstract data types (Sec. 4-intro, 4.1, 4.8–4.9)
- recursive algorithms (5.1)
- binary trees (5.4–5.7)
- sorting (6.1–6.4)
- binary search (12.4)
- binary search trees (12.5)
- probability and expectations (Goodrich & Tamassia, Section 1.3.4)

# Outline

# Problems (terminology)

First, we must introduce terminology so that we can precisely characterize what we mean by efficiency.

**Problem:** Given a problem instance, carry out a particular computational task.

**Problem Instance:** *Input* for the specified problem.

**Problem Solution:** *Output* (correct answer) for the specified problem instance.

**Size of a problem instance:** $Size(I)$ is a positive integer which is a measure of the size of the instance $I$.

Example: Sorting problem

# Algorithms and Programs

**Algorithm:** An algorithm is a *step-by-step process* (e.g., described in pseudocode) for carrying out a series of computations, given an arbitrary problem instance $I$.

**Algorithm solving a problem:** An Algorithm $A$ *solves* a problem $\Pi$ if, for every instance $I$ of $\Pi$, $A$ finds (computes) a valid solution for the instance $I$ in finite time.

**Program:** A program is an *implementation* of an algorithm using a specified computer language.

In this course, our emphasis is on algorithms (as opposed to programs or programming).

# Algorithms and Programs

**Pseudocode:** a method of communicating an algorithm to another person.

In contrast, a program is a method of communicating an algorithm to a computer.

Pseudocode

- omits obvious details, e.g. variable declarations,
- has limited if any error detection,
- sometimes uses English descriptions,
- sometimes uses mathematical notation.

# Algorithms and Programs

For a problem $\Pi$, we can have several algorithms.

For an algorithm $\mathcal{A}$ solving $\Pi$, we can have several programs (implementations).

Algorithms in practice: Given a problem $\Pi$

1. Design an algorithm $\mathcal{A}$ that solves $\Pi$. $\rightarrow$ **Algorithm Design**
2. Assess *correctness* and *efficiency* of $\mathcal{A}$. $\rightarrow$ **Algorithm Analysis**
3. If acceptable (correct and efficient), implement $\mathcal{A}$.

# Outline

# Efficiency of Algorithms/Programs

- How do we decide which algorithm or program is the most efficient solution to a given problem?
- In this course, we are primarily concerned with the *amount of time* a program takes to run. $\rightarrow$ **Running Time**
- We also may be interested in the *amount of memory* the program requires. $\rightarrow$ **Space**
- The amount of time and/or memory required by a program will depend on *Size(I)*, the size of the given problem instance *I*.

# Running Time of Algorithms/Programs

First Option: *experimental studies*

- Write a program implementing the algorithm.
- Run the program with inputs of varying size and composition.
- Use a method like `clock()` (from `time.h`) to get an accurate measure of the actual running time.
- Plot/compare the results.

# Running Time of Algorithms/Programs

Shortcomings of experimental studies

- Implementation may be complicated/costly.
- Timings are affected by many factors: *hardware* (processor, memory), *software environment* (OS, compiler, programming language), and human factors (programmer).
- We cannot test all inputs; what are good *sample inputs*?
- We cannot easily compare two algorithms/programs.

We want a framework that:

- Does not require implementing the algorithm.
- Is independent of the hardware/software environment.
- Takes into account all input instances.

We need some *simplifications*.

# Overview of Algorithm Analysis

We will develop several aspects of algorithm analysis in the next slides.

- Algorithms are presented in structured high-level *pseudocode* which is language-independent.
- Analysis of algorithms is based on an *idealized computer model*.
- The efficiency of an algorithm (with respect to time) is measured in terms of its *growth rate* (this is called the *complexity* of the algorithm).

# Running Time Simplifications

**Overcome dependency on hardware/software**

- Express algorithms using *pseudo-code*
- Instead of time, count the number of *primitive operations*
- Implicit assumption: primitive operations have fairly similar, though different, running time on different systems

**Random Access Machine (RAM)** Model:

- The *random access machine* has a set of memory cells, each of which stores one item (word) of data.
- Any *access to a memory location* takes constant time.
- Any *primitive operation* takes constant time.
- The *running time* of a program can be computed to be the number of memory accesses plus the number of primitive operations.

This is an idealized model, so these assumptions may not be valid for a "real" computer.

# Running Time Simplifications

**Simplify Comparisons**

- Example: Compare $100n$ with $10n^2$
- Idea: Use *order notation*
- Informally: ignore constants and lower order terms

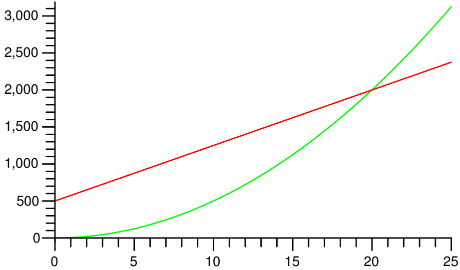
We will simplify our analysis by considering the behaviour of algorithms for large inputs sizes.

# Outline

# Order Notation

*O*-**notation:** $f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 > 0$ such that $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$.

Example: $f(n) = 75n + 500$ and $g(n) = 5n^2$ (e.g. $c = 1, n_0 = 20$)




**Note:** The absolute value signs in the definition are irrelevant for analysis of run-time or space, but are useful in other applications of asymptotic notation.

# Example of Order Notation

In order to prove that $2n^2 + 3n + 11 \in O(n^2)$ from first principles, we need to find $c$ and $n_0$ such that the following condition is satisfied:

$$0 \leq 2n^2 + 3n + 11 \leq c\, n^2 \text{ for all } n \geq n_0.$$

note that not all choices of $c$ and $n_0$ will work.

# Asymptotic Lower Bound

- We have $2n^2 + 3n + 11 \in O(n^2)$.
- But we also have $2n^2 + 3n + 11 \in O(n^{10})$.
- We want a **tight** asymptotic bound.

**$\Omega$-notation:** $f(n) \in \Omega(g(n))$ if there exist constants $c > 0$ and $n_0 > 0$ such that $c\,|g(n)| \leq |f(n)|$ for all $n \geq n_0$.

**$\Theta$-notation:** $f(n) \in \Theta(g(n))$ if there exist constants $c_1, c_2 > 0$ and $n_0 > 0$ such that $c_1\,|g(n)| \leq |f(n)| \leq c_2\,|g(n)|$ for all $n \geq n_0$.

$$f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n))$$

# Example of Order Notation

Prove that $\frac{1}{2}n^2 - 5n \in \Omega(n^2)$ from first principles.

# Strictly smaller/larger asymptotic bounds

- We have $f(n) = 2n^2 + 3n + 11 \in \Theta(n^2)$.
- How to express that $f(n)$ is **asymptotically strictly smaller** than $n^3$?

*o*-**notation:** $f(n) \in o(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $|f(n)| < c\,|g(n)|$ for all $n \geq n_0$.

$\omega$-**notation:** $f(n) \in \omega(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq c\,|g(n)| < |f(n)|$ for all $n \geq n_0$.

- Rarely proved from first principles.

# Relationships between Order Notations

- $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$
- $f(n) \in o(g(n)) \Leftrightarrow g(n) \in \omega(f(n))$

- $f(n) \in o(g(n)) \Rightarrow f(n) \in O(g(n))$
- $f(n) \in o(g(n)) \Rightarrow f(n) \notin \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \in \Omega(g(n))$
- $f(n) \in \omega(g(n)) \Rightarrow f(n) \notin O(g(n))$

# Algebra of Order Notations

**"Identity" rule:** $f(n) \in \Theta(f(n))$

**"Maximum" rules:** Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Then:

- $O(f(n) + g(n)) = O(\max\{f(n), g(n)\})$
- $\Omega(f(n) + g(n)) = \Omega(\max\{f(n), g(n)\})$

**Transitivity:**

- If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$ then $f(n) \in O(h(n))$.
- If $f(n) \in \Omega(g(n))$ and $g(n) \in \Omega(h(n))$ then $f(n) \in \Omega(h(n))$.

# Techniques for Order Notation

Suppose that $f(n) > 0$ and $g(n) > 0$ for all $n \geq n_0$. Suppose that

$$L = \lim_{n\to\infty} \frac{f(n)}{g(n)}.$$

Then

$$f(n) \in \begin{cases} o(g(n)) & \text{if } L = 0 \\ \Theta(g(n)) & \text{if } 0 < L < \infty \\ \omega(g(n)) & \text{if } L = \infty. \end{cases}$$

The required limit can often be computed using *l'Hôpital's rule*. Note that this result gives *sufficient* (but not necessary) conditions for the stated conclusions to hold.

# Example 1

Let $f(n)$ be a polynomial of degree $d \geq 0$:

$$f(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_1 n + c_0$$

for some $c_d > 0$.

Then $f(n) \in \Theta(n^d)$:

# Example 2

Prove that $n(2 + \sin n\pi/2)$ is $\Theta(n)$. Note that $\lim_{n\to\infty}(2 + \sin n\pi/2)$ does not exist.

# Growth Rates

- If $f(n) \in \Theta(g(n))$, then the *growth rates* of $f(n)$ and $g(n)$ are the *same*.
- If $f(n) \in o(g(n))$, then we say that the growth rate of $f(n)$ is *less than* the growth rate of $g(n)$.
- If $f(n) \in \omega(g(n))$, then we say that the growth rate of $f(n)$ is *greater than* the growth rate of $g(n)$.
- Typically, $f(n)$ may be "complicated" and $g(n)$ is chosen to be a very simple function.

# Example 3

Compare the growth rates of $\log n$ and $n$.

Now compare the growth rates of $(\log n)^c$ and $n^d$ (where $c > 0$ and $d > 0$ are arbitrary numbers).

# Common Growth Rates

Commonly encountered growth rates in analysis of algorithms include the following (in increasing order of growth rate):

- $\Theta(1)$ (*constant complexity*),
- $\Theta(\log n)$ (*logarithmic complexity*),
- $\Theta(n)$ (*linear complexity*),
- $\Theta(n \log n)$(*linearithmic*),
- $\Theta(n \log^k n)$, for some constant $k$ (*quasi-linear*),
- $\Theta(n^2)$ (*quadratic complexity*),
- $\Theta(n^3)$ (*cubic complexity*),
- $\Theta(2^n)$ (*exponential complexity*).

# How Growth Rates Affect Running Time

It is interesting to see how the running time is affected when the size of the problem instance **doubles** (i.e., $n \to 2n$).

- constant complexity: $T(n) = c$ $\rightsquigarrow$ $T(2n) = c$.
- logarithmic complexity: $T(n) = c \log n$ $\rightsquigarrow$ $T(2n) = T(n) + c$.
- linear complexity: $T(n) = cn$ $\rightsquigarrow$ $T(2n) = 2T(n)$.
- linearithmic $\Theta(n \log n)$: $T(n) = cn \log n$ $\rightsquigarrow$ $T(2n) = 2T(n) + 2cn$.
- quadratic complexity: $T(n) = cn^2$ $\rightsquigarrow$ $T(2n) = 4T(n)$.
- cubic complexity: $T(n) = cn^3$ $\rightsquigarrow$ $T(2n) = 8T(n)$.
- exponential complexity: $T(n) = c2^n$ $\rightsquigarrow$ $T(2n) = (T(n))^2/c$.

# Outline

# Techniques for Algorithm Analysis

- Goal: Use asymptotic notation to simplify run-time analysis.
- Running time of an algorithm depends on the *input size* $n$.

```
Test1(n)
    sum ← 0
    for i ← 1 to n do
        for j ← i to n do
            sum ← sum + (i − j)^2
    return sum
```

- Identify *elementary operations* that require $\Theta(1)$ time.
- The complexity of a loop is expressed as the *sum* of the complexities of each iteration of the loop.
- Nested loops: start with the innermost loop and proceed outwards. This gives *nested summations*.

# Techniques for Algorithm Analysis

Two general strategies are as follows.

- Use Θ-bounds *throughout the analysis* and obtain a Θ-bound for the complexity of the algorithm.
- Prove a $O$-bound and a *matching* Ω-bound *separately*.
  Use upper bounds (for $O$-bounds) and lower bounds (for Ω-bound) early and frequently.
  This may be easier because upper/lower bounds are easier to sum.

```
Test2(A, n)
1.   max ← 0
2.   for i ← 1 to n do
3.       for j ← i to n do
4.           sum ← 0
5.           for k ← i to j do
6.               sum ← A[k]
7.   return max
```

# Complexity of Algorithms

- Algorithm can have different running times on two instances of the same size.

```
Test3(A, n)
A: array of size n
    for i ← 1 to n − 1 do
        j ← i
        while j > 0 and A[j] > A[j − 1] do
            swap A[j] and A[j − 1]
            j ← j − 1
```

Let $T_A(I)$ denote the running time of an algorithm $A$ on instance $I$.

**Worst-case complexity of an algorithm**: take the worst $I$

**Average-case complexity of an algorithm**: average over $I$

# Complexity of Algorithms

**Worst-case complexity of an algorithm:** The worst-case running time of an algorithm $A$ is a function $f : \mathbb{Z}^+ \to \mathbb{R}$ mapping $n$ (the input size) to the *longest* running time for any input instance of size $n$:

$$T_A(n) = \max\{T_A(I) : Size(I) = n\}.$$

**Average-case complexity of an algorithm:** The average-case running time of an algorithm $A$ is a function $f : \mathbb{Z}^+ \to \mathbb{R}$ mapping $n$ (the input size) to the *average* running time of $A$ over all instances of size $n$:

$$T_A^{avg}(n) = \frac{1}{|\{I : Size(I) = n\}|} \sum_{\{I : Size(I) = n\}} T_A(I).$$

# O-notation and Complexity of Algorithms

- It is important not to try and make *comparisons* between algorithms using O-notation.
- For example, suppose algorithm $A_1$ and $A_2$ both solve the same problem, $A_1$ has worst-case run-time $O(n^3)$ and $A_2$ has worst-case run-time $O(n^2)$.
- Observe that we *cannot* conclude that $A_2$ is more efficient than $A_1$ for all input!
  1. The worst-case run-time may only be achieved on some instances.
  2. O-notation is an upper bound. $A_1$ may well have worst-case run-time $O(n)$. If we want to be able to compare algorithms, we should always use $\Theta$-notation.

# Outline

# Design of MergeSort

**Input:** Array $A$ of $n$ integers

- *Step 1:* We split $A$ into two subarrays: $A_L$ consists of the first $\lceil \frac{n}{2} \rceil$ elements in $A$ and $A_R$ consists of the last $\lfloor \frac{n}{2} \rfloor$ elements in $A$.
- *Step 2:* *Recursively* run *MergeSort* on $A_L$ and $A_R$.
- *Step 3:* After $A_L$ and $A_R$ have been sorted, use a function *Merge* to merge them into a single sorted array.

# MergeSort

To avoid copying sub-arrays, the recursion uses parameters that indicate the range of the array that needs to be sorted.

*MergeSort*$(A, \ell \leftarrow 0, r \leftarrow n-1)$
$A$: array of size $n$, $0 \leq \ell \leq r \leq n-1$
    **if** $(r \leq \ell)$ **then**
        return
    **else**
        $m = (r + \ell)/2$
        *MergeSort*$(A, \ell, m)$
        *MergeSort*$(A, m+1, r)$
        *Merge*$(A, \ell, m, r)$

# Merge

*Merge*$(A, \ell, m, r)$

$A[0..n-1]$ is an array, $A[\ell..m]$ is sorted, $A[m+1..r]$ is sorted

    initialize auxiliary array $S[0..n-1]$
    copy $A[\ell..r]$ into $S[\ell..r]$
    int $i_L \leftarrow \ell$; int $i_R \leftarrow m+1$;
    **for** $(k \leftarrow \ell; k \leq r; k{+}{+})$ **do**
        **if** $(i_L > m)$ $A[k] \leftarrow S[i_R{+}{+}]$
        **elsif** $(i_R > r)$ $A[k] \leftarrow S[i_L{+}{+}]$
        **elsif** $(S[i_L] \leq S[i_R])$ $A[k] \leftarrow S[i_L{+}{+}]$
        **else** $A[k] \leftarrow S[i_R{+}{+}]$

Sedgewick's code is slightly more complicated to avoid having to check whether $i_R$ and $i_L$ are out-of-boundary.

*Merge* takes time $\Theta(r - l + 1)$, i.e., $\Theta(n)$ time for merging $n$ elements.

# Analysis of MergeSort

Let $T(n)$ denote the time to run *MergeSort* on an array of length $n$.

- Step 1 takes time $\Theta(n)$
- Step 2 takes time $T\left(\lceil \frac{n}{2} \rceil\right) + T\left(\lfloor \frac{n}{2} \rfloor\right)$
- Step 3 takes time $\Theta(n)$

The *recurrence relation* for $T(n)$ is as follows:

$$T(n) = \begin{cases} T\left(\lceil \frac{n}{2} \rceil\right) + T\left(\lfloor \frac{n}{2} \rfloor\right) + \Theta(n) & \text{if } n > 1 \\ \Theta(1) & \text{if } n = 1. \end{cases}$$

It suffices to consider the following *exact recurrence*, with constant factor $c$ replacing $\Theta$'s:

$$T(n) = \begin{cases} T\left(\lceil \frac{n}{2} \rceil\right) + T\left(\lfloor \frac{n}{2} \rfloor\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

# Analysis of MergeSort

- The following is the corresponding *sloppy recurrence* (it has floors and ceilings removed):

$$T(n) = \begin{cases} 2\,T\left(\frac{n}{2}\right) + cn & \text{if } n > 1 \\ c & \text{if } n = 1. \end{cases}$$

- The exact and sloppy recurrences are *identical* when $n$ is a power of 2.
- The recurrence can easily be solved by various methods when $n = 2^j$. The solution has growth rate $T(n) \in \Theta(n \log n)$.
- It is possible to show that $T(n) \in \Theta(n \log n)$ *for all n* by analyzing the exact recurrence.

# Some Recurrence Relations

| Recursion | resolves to | example |
|---|---|---|
| $T(n) = T(n/2) + \Theta(1)$ | $T(n) \in \Theta(\log n)$ | Binary search |
| $T(n) = 2T(n/2) + \Theta(n)$ | $T(n) \in \Theta(n \log n)$ | Mergesort |
| $T(n) = 2T(n/2) + \Theta(\log n)$ | $T(n) \in \Theta(n)$ | Heapify ($\rightarrow$ later) |
| $T(n) = T(cn) + \Theta(n)$ for some $0 < c < 1$ | $T(n) \in \Theta(n)$ | Selection ($\rightarrow$ later) |
| $T(n) = 2T(n/4) + \Theta(1)$ | $T(n) \in \Theta(\sqrt{n})$ | Range Search ($\rightarrow$ later) |
| $T(n) = T(\sqrt{n}) + \Theta(1)$ | $T(n) \in \Theta(\log \log n)$ | Interpolation Search ($\rightarrow$ later) |

- Once you know the result, it is (usually) easy to prove by induction.
- Many more recursions, and some methods to find the result, in cs341.

# Outline

# Order Notation Summary

$O$-**notation:** $f(n) \in O(g(n))$ if there exist constants $c > 0$ and $n_0 > 0$ such that $|f(n)| \leq c\,|g(n)|$ for all $n \geq n_0$.

$\Omega$-**notation:** $f(n) \in \Omega(g(n))$ if there exist constants $c > 0$ and $n_0 > 0$ such that $c\,|g(n)| \leq |f(n)|$ for all $n \geq n_0$.

$\Theta$-**notation:** $f(n) \in \Theta(g(n))$ if there exist constants $c_1, c_2 > 0$ and $n_0 > 0$ such that $c_1\,|g(n)| \leq |f(n)| \leq c_2\,|g(n)|$ for all $n \geq n_0$.

$o$-**notation:** $f(n) \in o(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $|f(n)| < c\,|g(n)|$ for all $n \geq n_0$.

$\omega$-**notation:** $f(n) \in \omega(g(n))$ if for all constants $c > 0$, there exists a constant $n_0 > 0$ such that $c\,|g(n)| < |f(n)|$ for all $n \geq n_0$.

# Useful Sums

**Arithmetic sequence:**

$\sum_{i=0}^{n-1} i = ???$ $\qquad$ $\sum_{i=0}^{n-1}(a+di) = na + \frac{dn(n-1)}{2} \in \Theta(n^2) \quad \text{if } d \neq 0.$

**Geometric sequence:**

$\sum_{i=0}^{n-1} r^i = ???$ $\qquad$ $\sum_{i=0}^{n-1} a\, r^i = \begin{cases} a\dfrac{r^n-1}{r-1} & \in \Theta(r^n) \quad \text{if } r > 1 \\ na & \in \Theta(n) \quad \text{if } r = 1 \\ a\dfrac{1-r^n}{1-r} & \in \Theta(1) \quad \text{if } 0 < r < 1. \end{cases}$

**Harmonic sequence:**

$\sum_{i=1}^{n} \frac{1}{i} = ???$ $\qquad$ $H_n := \sum_{i=1}^{n} \frac{1}{i} = \ln n + \gamma + o(1) \in \Theta(\log n)$

**A few more:**

$\sum_{i=1}^{n} \frac{1}{i^2} = ???$ $\qquad$ $\sum_{i=1}^{n} \frac{1}{i^2} = \frac{\pi^2}{6} \in \Theta(1)$

$\sum_{i=1}^{n} i^k = ???$ $\qquad$ $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1}) \quad \text{for } k \geq 0$

# Useful Math Facts

**Logarithms:**

- $c = \log_b(a)$ means $b^c = a$. E.g. $n = 2^{\log n}$.
- $\log(a)$ (in this course) means $\log_2(a)$
- $\log(a \cdot c) = \log(a) + \log(c)$, $\log(a^c) = c\log(a)$,
- $\log_b(a) = \frac{\log_c a}{\log_c b} = \frac{1}{\log_a(b)}$.
- $a^{\log_b c} = c^{\log_b a}$
- $\ln(x) =$ natural log $= \log_e(x)$, $\frac{d}{dx}\ln x = \frac{1}{x}$

**Factorial:**

- $n! := n(n-1)(n-2)\cdot \cdots \cdot 2 \cdot 1 = \#$ ways to permute $n$ elements
- $\log(n!) = \log n + \log(n-1) + \cdots + \log 2 + \log 1 \in \Theta(n \log n)$

**Probability and moments:**

- $E[aX] = aE[X]$, $E[X+Y] = E[X] + E[Y]$ (linearity of expectation)