# CS 240 – Data Structures and Data Management

## Module 6: Dictionaries for special keys

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References: Sedgewick 12.4, 15.2-15.4
Goodrich & Tamassia 9.2.1-9.2.2

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# Outline

1. Lower bound
2. Interpolation Search
3. Tries
   - Standard Tries
   - Variations of Tries
   - Compressed Tries

# Lower bound for search

The fastest implementations of the dictionary ADT require $\Theta(\log n)$ time to search a dictionary containing $n$ items. Is this the best possible?

Theorem: In the comparison model (on the keys),
$\Omega(\log n)$ comparisons are required to search a size-$n$ dictionary.

Proof:

But can we beat the lower bound for special keys?

# Interpolation Search: Motivation

Ordered array

- *insert, delete*: $\Theta(n)$
- *search*: $\Theta(\log n)$

binary search$(A[\ell, r], k)$: Compare at index $\lfloor \frac{\ell + r}{2} \rfloor = \ell + \lfloor \frac{1}{2}(r - \ell) \rfloor$

| | $\ell$ | | $\downarrow$ | $\downarrow$ | $r$ | |
|---|---|---|---|---|---|---|
| | 40 | | | | 120 | |

Question: If keys are numbers, where would you expect key $k = 100$?

Interpolation Search$(A[\ell, r], k)$: Compare at index $\ell + \left\lfloor \frac{k - A[\ell]}{A[r] - A[\ell]}(r - \ell) \right\rfloor$

# Interpolation Search Example

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 449 | 450 | 600 | 800 | 1000 | 1200 | 1500 |

*Search*(449):

- Initially $\ell = 0$, $r = n - 1 = 10$, $m = \ell + \lfloor \frac{449-0}{1500-0}(10-0) \rfloor = \ell + 2 = 2$
- $\ell = 3$, $r = 10$, $m = \ell + \lfloor \frac{449-3}{1500-3}(10-3) \rfloor = \ell + 2 = 5$
- $\ell = 3$, $r = 4$, $m = \ell + \lfloor \frac{449-3}{449-3}(4-3) \rfloor = \ell + 1 = 4$, found at $A[4]$

Works well if keys are uniformly distributed:

- Can show: the array in which we recurse into has expected size $\sqrt{n}$.
- Recurrence relation is $T^{(\text{avg})}(n) = T^{(\text{avg})}(\sqrt{n}) + \Theta(1)$.
- This resolves to $T^{(\text{avg})}(n) \in \Theta(\log \log n)$.

But: Worst case performance $\Theta(n)$

# Interpolation Search

- Code very similar to binary search, but compare at interpolated index
- Need a few extra tests to avoid crash due to $A[\ell] = A[r]$

*Interpolation-search*$(A, n, k)$
$A$: Array of size $n$, $k$: key

$\ell \leftarrow 0$
$r \leftarrow n - 1$
**while** $((A[r] != A[\ell]) \&\& (k \geq A[\ell]) \&\& (k \leq A[r]))$
    $m \leftarrow \ell + \lfloor \frac{k - A[\ell]}{A[r] - A[\ell]} \cdot (r - \ell) \rfloor$
    **if** $(A[m] < k)$ $\ell = m + 1$
    **elsif** $(k < A[m])$ $r = m - 1$
    **else** return $m$
**if** $(k = A[\ell])$ return $\ell$
**else** return "not found"

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# Tries: Introduction

- **Trie (Radix Tree)**: A dictionary for binary strings
  - Comes from retrieval, but pronounced "try"
  - A tree based on **bitwise comparisons**
  - Similar to **radix sort**: use individual bits, not the whole key
- Keys can have different number of bits

**Prefix** of a string $S[0..n-1]$: a substring $S[0..i]$ of $S$ for some $0 \leq i \leq n-1$.

**Prefix-free**: there is no pair of binary strings in the dictionary where one is the prefix of the other.

Assumption: Dictionary is prefix-free:

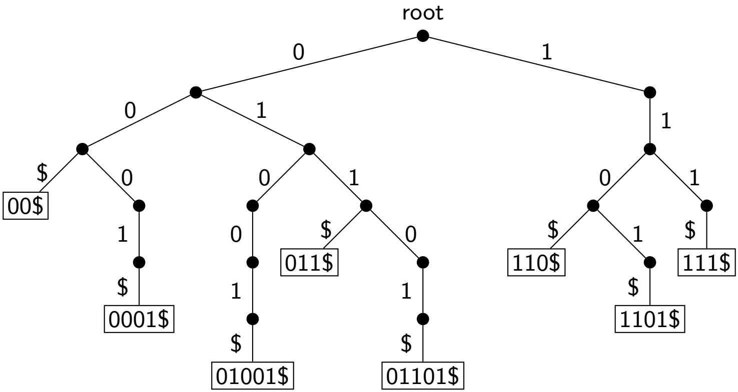
- This is always satisfied if all strings have the same length.
- This is always satisfied if all strings end with a special 'end-of-word' character \$.

# Tries: structure

Structure of trie:

- Items (keys) are stored **only** in the leaf nodes
- Edge to child is labelled with corresponding bit or \$

Example: A trie for
$S = \{00\$, 0001\$, 01001\$, 011\$, 01101\$, 110\$, 1101\$, 111\$\}$



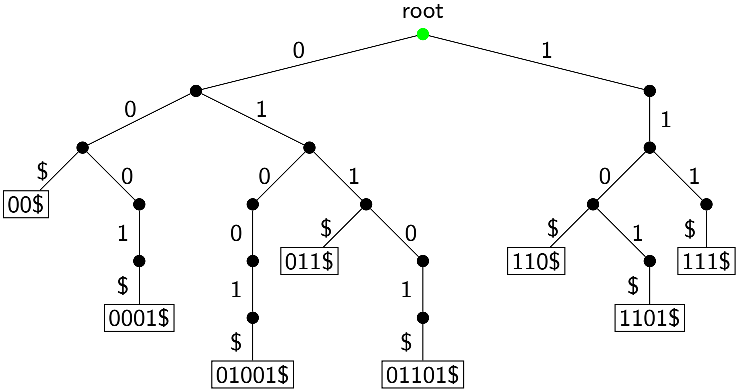
# Tries: Search

- start from the root and the most significant bit of $x$
- follow the link that corresponds to the current bit in $x$; return failure if the link is missing
- return success if we reach a leaf (it must store $x$)
- else recurse on the new node and the next bit of $x$

```
Trie-search(v ← root, d ← 0, x)
v: node of trie; d: level of v, x: word
    if v is a leaf
        return v
    else
        let c be child of v labelled with x[d]
        if there is no such child
            return "not found"
        else Trie-search(c, d + 1, x)
```

# Tries: Search Example

Example: Search(011$)

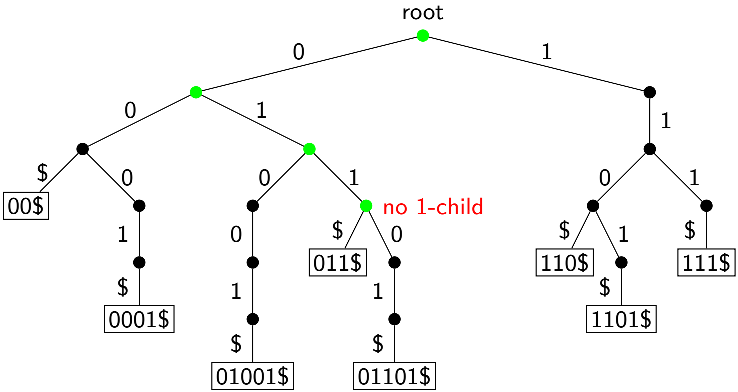


# Tries: Insert & Delete

- **Insert($x$)**
  - Search for $x$, this should be unsuccessful
  - Suppose we finish at a node $v$ that is missing a suitable child.
    Note: $x$ has extra bits left.
  - Expand the trie from the node $v$ by adding necessary nodes that correspond to extra bits of $x$.
- **Delete($x$)**
  - Search for $x$
  - let $v$ be the leaf where $x$ is found
  - delete $v$ and all ancestors of $v$ until we reach an ancestor that has two children.
- Time Complexity of all operations: $\Theta(|x|)$
  $|x|$: length of binary string $x$, i.e., the number of bits in $x$

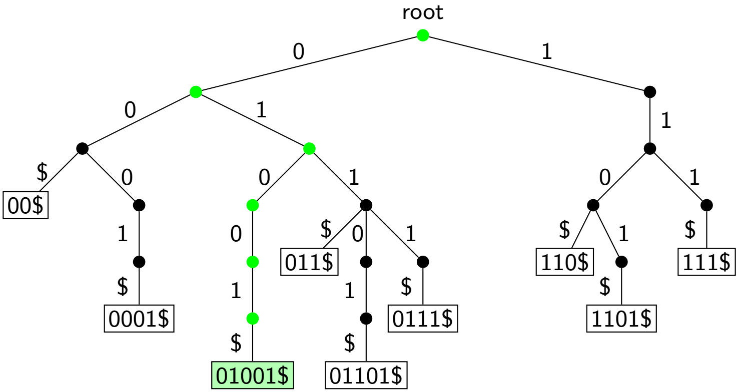
# Tries: Insert Example

Example: Insert(0111$)



# Tries: Delete Example

Example: Delete(01001$)



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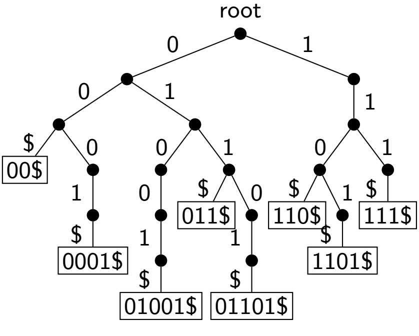
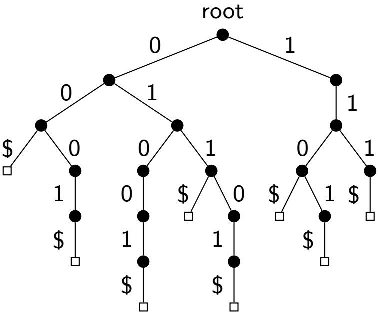
# Variation 1 of Tries: No leaf labels

Do not store actual keys at the leaves.

- The key is stored implicitly through the characters along the path to the leaf. It therefore need not be stored again.
- This halves the amount of space needed.

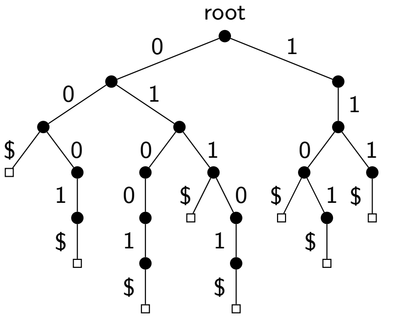


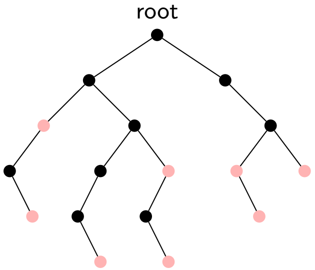



# Variation 2 of Tries: Allow Proper Prefixes

Allow prefixes to be in dictionary.

- Then internal nodes may also represent keys. We then use a *flag* at each node to indicate whether this represents a key of the dictionary.
- This replaces the reference to the \$-child with a flag (one bit) and saves space.
- For bitstrings, we then do not need \$, and so every node has at most two children. Can express 0-child and 1-child implicitly via left and right child.



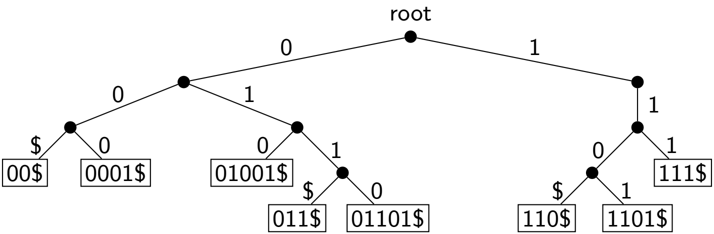



# Variations 3 of Tries: Remove Chains to Labels

Stop adding nodes to trie as soon as the key is unique.

- A node has a child only if it has at least two descendants.
- Saves space if there are only few bitstrings that are long.
- Note that this variation *cannot* be combined with the the first variation (why not?)

This variation is the one presented in Sedgewick.



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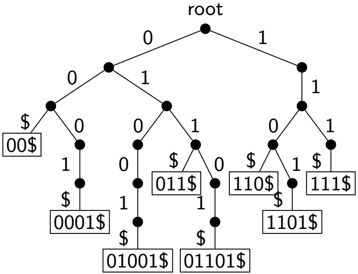
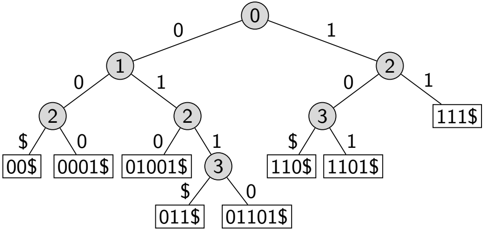
# Compressed Tries (Patricia Tries)

- Morrison (1968):
  *Patricia: Practical Algorithm to Retrieve Information Coded in Alphanumeric*
- Compress paths of nodes with only one child
- Each node stores an *index* indicating the next bit to be tested during a search (index $= 0$ for the first bit, index $= 1$ for the second bit, etc.)
- A compressed trie storing $n$ keys always has at most $n - 1$ internal (non-leaf) nodes

# Compressed Tries Example

Example: A trie and the equivalent compressed trie.



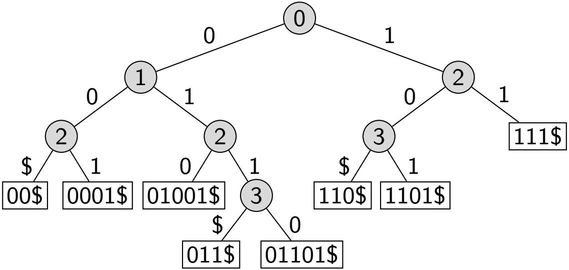
# Compressed Tries: Search

- start from the root and the bit indicated at that node
- follow the link that corresponds to the current bit in $x$; return failure if the link is missing
- if we reach a leaf, explicitly check whether word stored at leaf is $x$
- else recurse on the new node and the next bit of $x$

```
Patricia-Trie-search(v ← root, x)
v: node of trie; x: word
    if v is a leaf
        return strcmp(x, key(v))
    else
        let d be the bit stored at v
        let c be child of v labelled with x[d]
        if there is no such child
            return "not found"
        else Patricia-Trie-search(c, x)
```

# Compressed Tries: Search Example

Example: Search(10$) **unsuccessful**



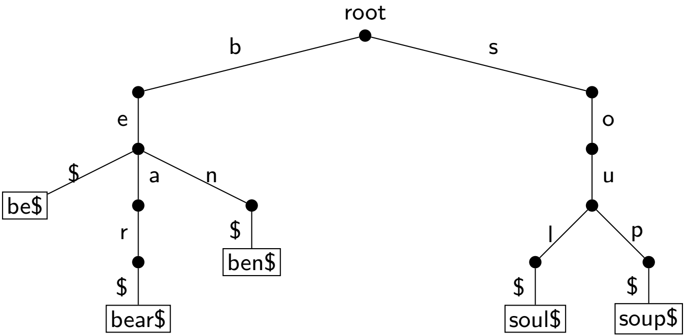
# Compressed Tries: Insert & Delete

- **Delete($x$)**:
  - Perform Search($x$)
  - Remove the node $v$ that stored $x$
  - Compress along path to $v$ whenever possible.
- **Insert($x$)**:
  - Perform Search($x$)
  - Let $v$ be the node where the search ended.
  - Conceptually simplest approach:
    - ★ Uncompress path from root to $v$.
    - ★ Insert $x$ as in an uncompressed trie.
    - ★ Compress paths from root to $v$ and from root to $x$.

    But it can also be done by only adding those nodes that are needed, see the textbook for details.
- All operations take $O(|x|)$ time.

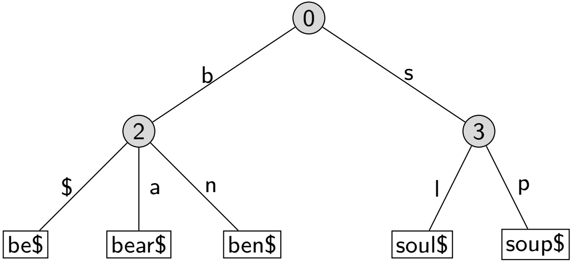
# Multiway Tries: Larger Alphabet

- To represent **Strings** over any **fixed alphabet** $\Sigma$
- Any node will have at most $|\Sigma| + 1$ children (one child for the end-of-word character \$)
- Example: A trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



# Compressed Multiway Tries

- **Compressed** multi-way tries
- Example: A compressed trie holding strings {bear\$, ben\$, be\$, soul\$, soup\$}



# Multiway Tries: Summary

- Operations Search($x$), Insert($x$) and Delete($x$) are exactly as for tries for bitstrings.
- Run-time $O(|x| \cdot (\text{time to find the appropriate child}))$
- Each node now has up to $|\Sigma| + 1$ references to children. How should they be stored?
  - Could store array of size $|\Sigma| + 1$ at each node. $O(1)$ time to find the appropriate child, but then for $n$ nodes the total space is $O(|\Sigma| n)$.
  - Could store list of children at each node. Then total space is $O(n)$, but time to find child increases to $O(|\Sigma|)$. Use MTF!
  - Could use some good dictionary implementation (AVL-tree? Skip list?) at each node. Then the total space is $O(n)$ and the time to find a child is $O(\log |\Sigma|)$ or better. Best in theory, but in practice not worth it unless the alphabet is huge.