CS 240 – Data Structures and Data Management

Module 11: External Memory

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Based on lecture notes by many previous cs240 instructors

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References: Goodrich & Tamassia 14.1, Sedgewick 16.4

Outline

1. External Memory
   - Motivation
   - External Dictionaries
   - 2-3 Trees
   - (a, b)-Trees
   - B-Trees
   - External sorting

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Different levels of memory

Current architectures:

- registers (very fast, very small)
- cache L1, L2 (still fast, less small)
- main memory
- external memory: disk or cloud (slow, very large)

General question: how to adapt our algorithms to take the memory hierarchy into account, avoiding transfers as much as possible?

Observation: Accessing a single location in *external memory* (e.g. hard disk) automatically loads a whole block (or "page").

New objective: revisit all ADTs/problems with the objective of minimizing page loads.

The External-Memory Model (EMM)

Two levels of memory:

- internal memory (CPU, . . .)
  - size M
  - fast access
- external memory
  - unlimited space
  - slow access
  - data moved to internal memory in *blocks* of B cells

Cost of computation:

"I/O operations" = # blocks transfered between internal and external memory

I/O operations are also called *page loads* or *block transfers*.

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Dictionaries in external memory

Tree-based dictionary implementations have poor *memory locality*:
If an operation accesses m nodes, then it must access m spaced-out memory locations.

- In an AVL tree, $\Theta(\log n)$ pages are loaded in the worst case.
- Better solution: do more in single node $\rightsquigarrow$ B-trees

- First consider special case of B-trees: *2-3 trees*
  - 2-3-trees would also be interesting for implementing ADT Dictionaries in main memory (may be even faster than AVL-trees)
  - We first analyze their performance in main memory, and then (for B-trees) in external memory.

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2-3 Trees

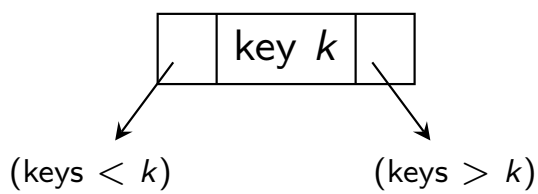
A 2-3 Tree is a balanced search tree that is not necessarily binary.

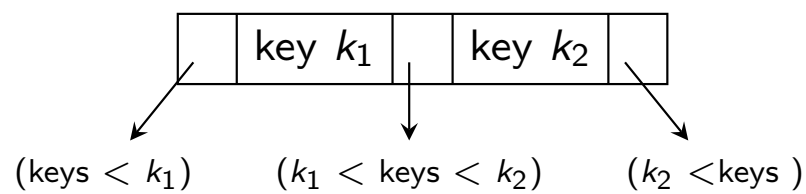
Structural properties:

- Every internal node is either
  - 1-node: *one KVP* and *two children*, or
  - 2-node: *two KVPs* and *three children*.
- The external nodes are *NIL* (do not store keys)
- All external nodes are at the same level.

Height-balance strictly enforced, but allow 2 types of nodes!

Order property: The keys at a node are between the keys in the subtrees.





2-3 Tree operations

Search: The order-property determines the subtree to search in.

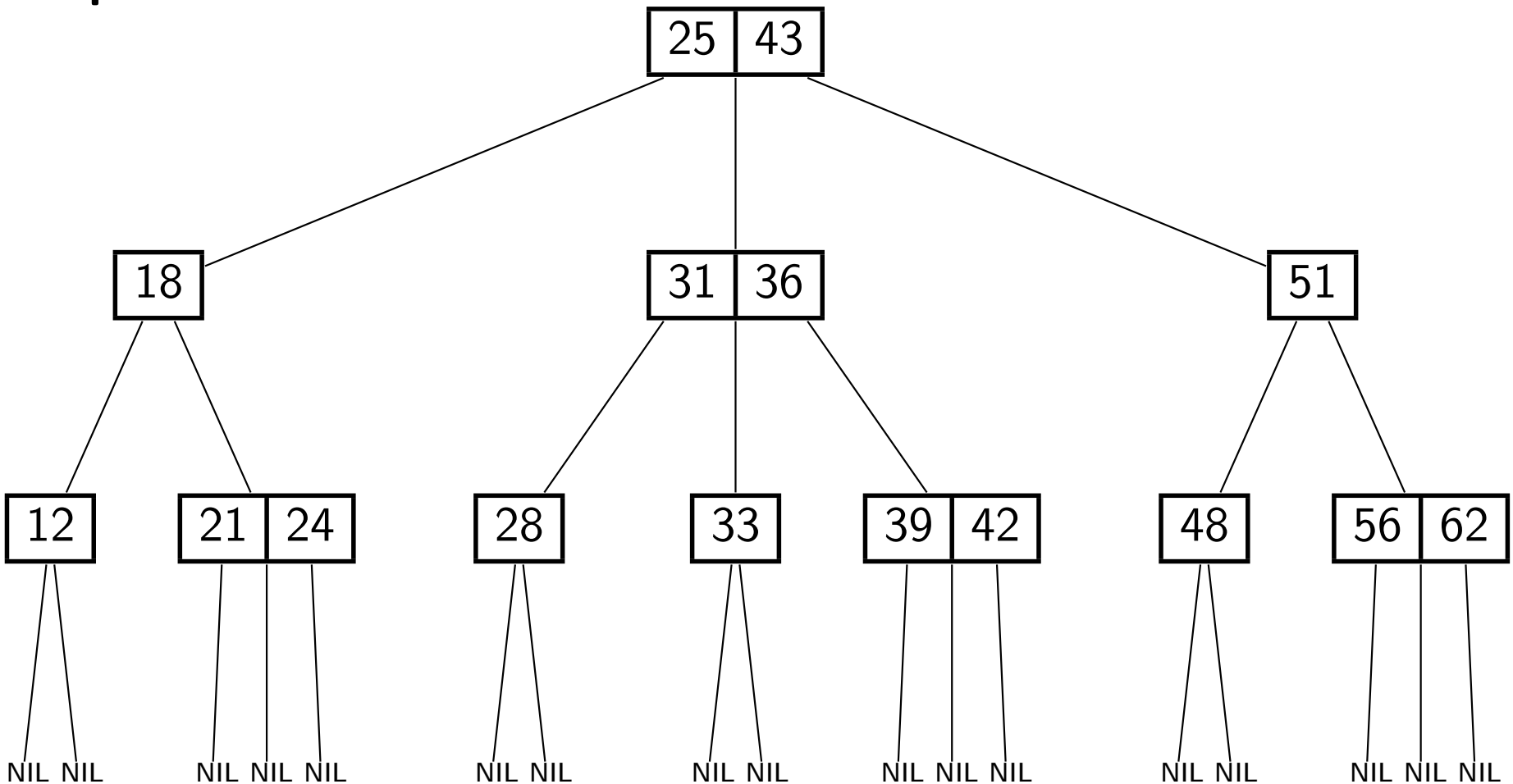
```
23TreeSearch(k, v ← root)
1.   Let c_0, k_1, ..., k_d, c_d be keys and children at v, in order
2.   if k ≥ k_1
3.       i ← maximal index such that k_i ≤ k
4.       if k_i = k return k_i
5.   else i ← 0
6.   23TreeSearch(k, c_i)
```

Insert: Nodes may grow from bottom to top.

- Search to find leaf ℓ where the new key k belongs.
- Add k and a NIL-child to ℓ. If ℓ now as 3 keys (**overflow**):
  - Split ℓ into two nodes ℓ, ℓ' with min and max key of ℓ
  - Move median key of ℓ into parent p of ℓ. Also make ℓ' child of p.
  - Recurse in p if it now has overflow.

Example: Insertion in a 2-3 tree

Example:

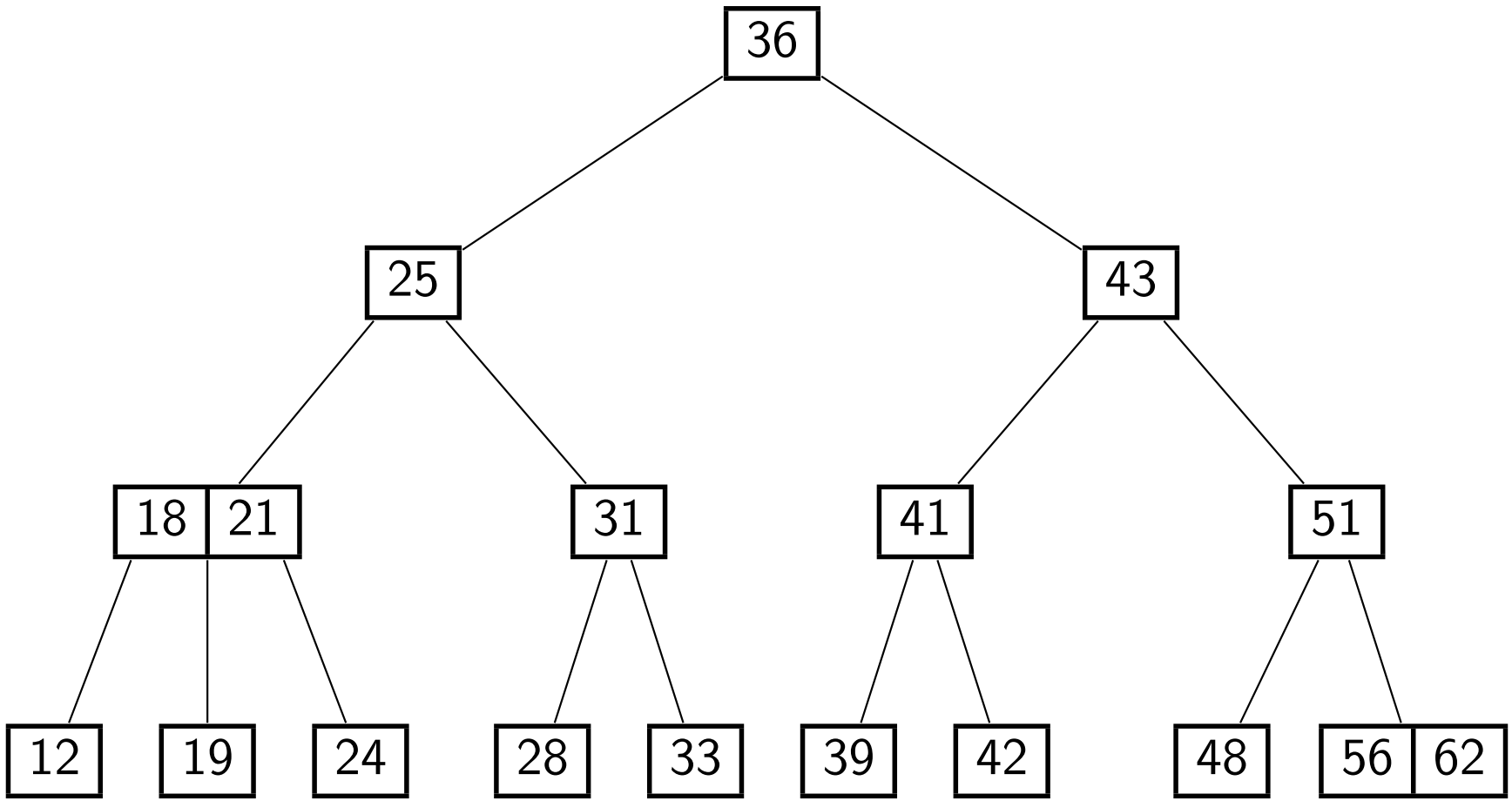


Deletion from a 2-3 Tree

- As with BSTs and AVL trees, we first swap the KVP k with its successor, so that it is now at a leaf ℓ.
- Delete k and one NIL-child from ℓ.
- If ℓ now has 0 keys (*underflow*)
  - If ℓ is the root, simply delete it. Else let p be the parent of ℓ.
  - If some *immediate* sibling u is a 2-node, perform a *transfer*:
    - ★ Find the key k_p in p that is between keys of ℓ and u.
    - ★ "Rotate:" move k_p into ℓ, move adjacent KVP from u into p, and re-arrange children suitably.
  - Otherwise, we *merge* ℓ and a 1-node sibling u:
    - ★ Find the key k_p in parent p between keys of ℓ and u.
    - ★ Combine ℓ and u into one node and move k_p into it.
    - ★ Recurse in p if it now has underflow.

2-3 Tree Deletion

Example:



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(a, b)-Trees

The 2-3 Tree is a specific type of (a, b)-tree:

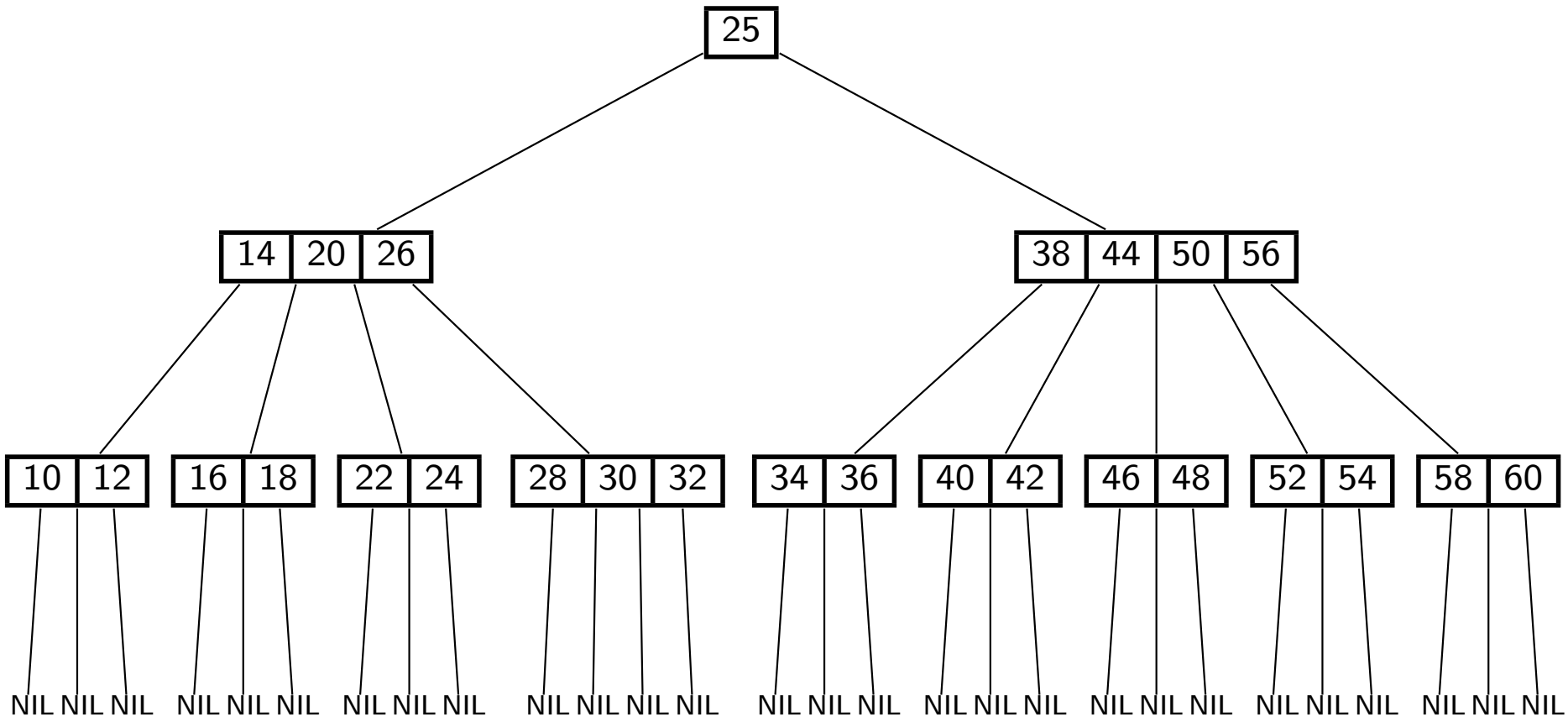
An (a, b)-*tree* satisfies:

- Each internal node has at least a children, unless it is the root. The root has at least 2 children.
- Each internal node has at most b children.
- If a node has k children, then it stores $k - 1$ key-value pairs (KVPs).
- External nodes store no keys and are at the same level.
- The keys in the node are between the keys in the corresponding children.

If $a \leq (b + 1)/2$, then *search*, *insert*, *delete* work just like for 2-3 trees, after re-defining underflow/overflow to consider the above constraints.

(a, b)-tree example

A $(3, 5)$-tree (it is also a valid $(3, 6)$-tree):



Height of an (a, b)-tree

What is the least number of KVPs in an (a, b)-tree of height-h?
(Height = # levels **not** counting the NIL-level -1)

Level	**Nodes** $\geq$	**Links/node** $\geq$	**KVP/node** $\geq$	**KVPs on level** $\geq$
0	1	2	1	1
1	2	a	$a-1$	$2(a-1)$
2	$2a$	a	$a-1$	$2a(a-1)$
3	$2a^2$	a	$a-1$	$2a^2(a-1)$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
h	$2a^{h-1}$	a	$a-1$	$2a^{h-1}(a-1)$

$$\text{Total: } n \geq 1 + 2(a-1)\sum_{i=0}^{h-1} a^i = 2a^h - 1$$

Therefore height of tree with n KVPs is $O(\log_a(n)) = O(\log(n)/\log(a))$.
Can also prove that the height is $\Omega(\log_b(n))$.

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B-trees

A *B-tree of order m* is a $(\lceil m/2 \rceil, m)$-tree.
A 2-3 tree is a B-tree of order 3.

Sedgewick uses M rather than m, but this is confusing since we set M to be the space in main memory.

Analysis (if entire B-tree is stored in main memory):

- *search*, *insert*, and *delete* each require $\Theta(\textit{height})$ node operations.
- Height is $\Theta(\log n / \log m)$.
- Assume each node stores its KVPs and child-pointers in a dictionary that supports $\Theta(\log m)$ search.

Total cost for search is $\Theta\left(\frac{\log n}{\log m} \cdot (\log m)\right) = \Theta(\log n)$. This is no better than 2-3-trees or AVL-trees.
Insert and delete need a bit more care.

Dictionaries in external memory

Main applications of B-trees: Store dictionaries in external memory.

Recall: In an AVL tree or 2-3 tree, $\Theta(\log n)$ pages are loaded in the worst case.

Instead, use a B-tree of order m, where m is chosen so that an m-node fits into a single page.

Each operation can be done with $\Theta(\text{height})$ page loads.

The height of a B-tree is $\Theta\big(\log n / \log m\big)$.

This results in huge savings of page loads.

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Sorting in external memory

Given an array A of n numbers, put them into sorted order.

Now assume n is huge and A is stored in blocks in external memory.

- Recall: Heapsort was optimal in time and space in RAM model
- **But:** Heapsort accesses A at indices that are far apart
  $\rightsquigarrow$ typically one page loads per array access.
- Mergesort adapts well to an array stored in external memory.
- It can be made even more effective using **d-way merge**: Merge d sorted runs into one sorted run.

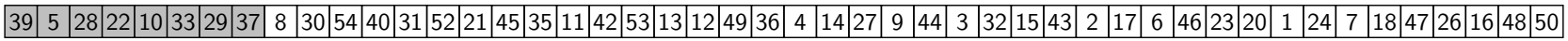
d-way merge

```
d-Way-Merge(S_1, ..., S_d)
S_1, ..., S_d are sorted sets (arrays/lists/stacks/queues)
 1.   P ← empty min-priority queue
 2.   S ← empty set
 3.   for i ← 1 to d do
 4.        P.insert((first element of S_i, i))
 5.   while P is not empty do
 6.        (x, i) ← deleteMin(P)
 7.        remove x from S_i and append it to S
 8.        if S_i is not empty do
 9.             P.insert((first element of S_i, i))
```

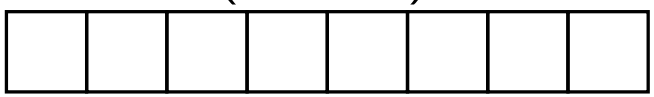
- Standard mergesort uses $d = 2$
- $d > 2$ could be used in internal memory as well, but the extra time to find minimum in the priority queue means the overall run-time is no better.

Mergesort in external memory

External ($B = 2$):



Internal ($M = 8$):



1. Create n/M sorted runs of length M. $\Theta(n/B)$ **IO-operations**
2. Merge the first $d \approx M/B - 1$ sorted runs using *d-Way-Merge*
3. Keep merging the next runs to reduce # runs by factor of d
   $\rightsquigarrow$ one round of merging. $\Theta(n/B)$ **IO-operations**
4. $\log_d(n/M)$ **rounds** of merging create sorted array.

Mergesort with external memory

Total page loads: $O(\log_d(n) \cdot n/B)$.

Assuming the EMM, one can prove lower bounds!

- $\Omega(\frac{n}{B})$ I/Os required to **scan** n elements.
- $\Omega(\frac{n}{B} \log_{M/B}(\frac{n}{B}))$ I/Os required to **sort** n elements with comparisons.
  - We don't prove that here.
- d-way Mergesort with $d \approx M/B$ is optimal (up to constant factors)!