CS 240 – Data Structures and Data Management

Module 2: Priority Queues

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Based on lecture notes by many previous cs240 instructors

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Outline

2. Priority Queues
   - Abstract Data Types
   - ADT Priority Queue
   - Binary Heaps
   - Operations in Binary Heaps
   - *PQ-sort* and *Heapsort*
   - Towards the Selection Problem

Outline

2 Priority Queues

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- ADT Priority Queue
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Abstract Data Types

Abstract Data Type (ADT): A description of *information* and a collection of *operations* on that information.

The information is accessed *only* through the operations.

We can have various **realizations** of an ADT, which specify:

- How the information is stored (**data structure**)
- How the operations are performed (**algorithms**)

Stack ADT

Stack: an ADT consisting of a collection of items with operations:

- *push*: inserting an item
- *pop*: removing (and typically returning) the most recently inserted item

Items are removed in LIFO (*last-in first-out*) order.
Items enter the stack at the *top* and are removed from the *top*.
We can have extra operations: *size*, *isEmpty*, and *top*

Applications: Addresses of recently visited web sites, procedure calls

Realizations of Stack ADT

- using arrays
- using linked lists

Queue ADT

Queue: an ADT consisting of a collection of items with operations:

- *enqueue*: inserting an item
- *dequeue*: removing (and typically returning) the least recently inserted item

Items are removed in FIFO (*first-in first-out*) order.
Items enter the queue at the *rear* and are removed from the *front*.
We can have extra operations: *size*, *isEmpty*, and *front*

Applications: Waiting lines, printer queues

Realizations of Queue ADT

- using (circular) arrays
- using linked lists

Outline

Priority Queue ADT

Priority Queue: An ADT consisting of a collection of items (each having a **priority**) with operations

- *insert*: inserting an item tagged with a priority
- *deleteMax*: removing and returning the item of *highest* priority

deleteMax is also called *extractMax* or *getmax*.
The priority is also called *key*.

The above definition is for a **maximum-oriented** priority queue. A **minimum-oriented** priority queue is defined in the natural way, replacing operation *deleteMax* by *deleteMin*,

Applications: typical "todo" list, simulation systems, sorting

Using a Priority Queue to Sort

PQ-Sort($A[0..n-1]$)
1. initialize PQ to an empty priority queue
2. **for** $i \leftarrow 0$ **to** $n-1$ **do**
3. $\quad PQ.insert(A[i])$
4. **for** $i \leftarrow n-1$ **down to** 0 **do**
5. $\quad A[i] \leftarrow PQ.deleteMax()$

- Note: Run-time depends on how we implement the priority queue.
- Sometimes written as: $O(initialization + n \cdot insert + n \cdot deleteMax)$

Realizations of Priority Queues

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- *insert*: $O(1)$
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Note: We assume **dynamic arrays**, i. e., expand by doubling as needed. (Amortized over all insertions this takes $O(1)$ extra time.)

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Realization 2: sorted arrays

- *insert*: $O(n)$
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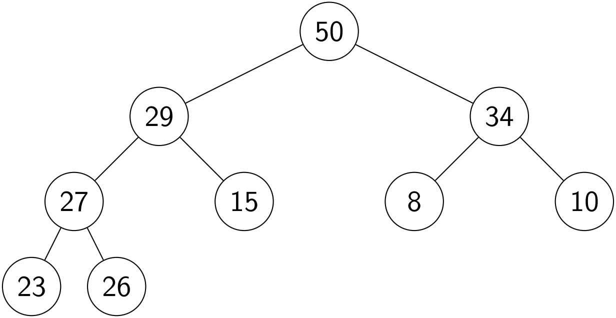
Realization 3: Heaps

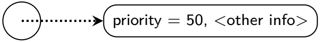
A **(binary) heap** is a certain type of binary tree.

You should know:

- A **binary tree** is either
  - empty, or
  - consists of three parts:
    a node and two binary trees (left subtree and right subtree).
- Terminology: root, leaf, parent, child, level, sibling, ancestor, descendant, etc.
- Any binary tree with n nodes has height at least $\log(n+1) - 1 \in \Omega(\log n)$.

Example Heap



(In our examples we only show the priorities, and we show them directly in the node. A more accurate picture would be )

Heaps – Definition

A **heap** is a binary tree with the following two properties:

1. **Structural Property:** All the levels of a heap are completely filled, except (possibly) for the last level. The filled items in the last level are *left-justified*.
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The full name for this is *max-oriented binary heap*.

Lemma: The height of a heap with n nodes is $\Theta(\log n)$.

Storing Heaps in Arrays

Heaps should *not* be stored as binary trees!

Let H be a heap of n items and let A be an array of size n. Store root in $A[0]$ and continue with elements *level-by-level* from top to bottom, in each level left-to-right.

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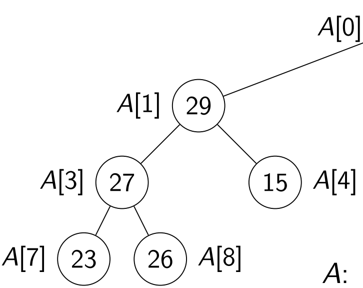

	0	1	2	3	4	5	6	7	8
A:	50	29	34	27	15	8	10	23	26

Heaps in Arrays – Navigation

It is easy to navigate the heap using this array representation:

- the *root* node is at index 0
  (We use "node" and "index" interchangeably in this implementation.)
- the *last* node is $n - 1$ (where n is the size)
- the *left child* of node i (if it exists) is node $2i + 1$
- the *right child* of node i (if it exists) is node $2i + 2$
- the *parent* of node i (if it exists) is node $\lfloor \frac{i-1}{2} \rfloor$
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We should hide implementation details using helper-functions!

- functions *root()*, *last()*, *parent(i)*, etc.

Some of these helper-functions need to know n (but we omit this in the code for simplicity).

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> *fix-up*(A, i)
> *i: an index corresponding to a node of the heap*
> 1. **while** $parent(i)$ exists **and** $A[parent(i)].key < A[i].key$ **do**
> 2. swap $A[i]$ and $A[parent(i)]$
> 3. $i \leftarrow parent(i)$

The new item "bubbles up" until it reaches its correct place in the heap.

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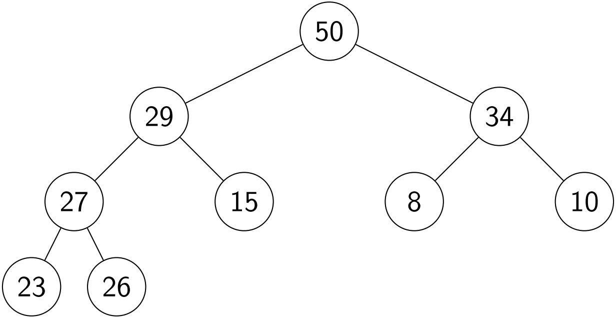
Time: $O(\text{height of heap}) = O(\log n)$.

fix-up example



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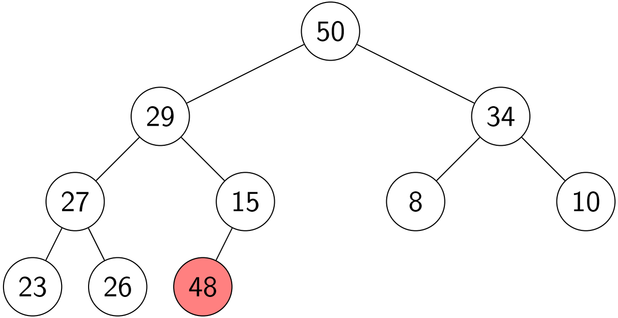

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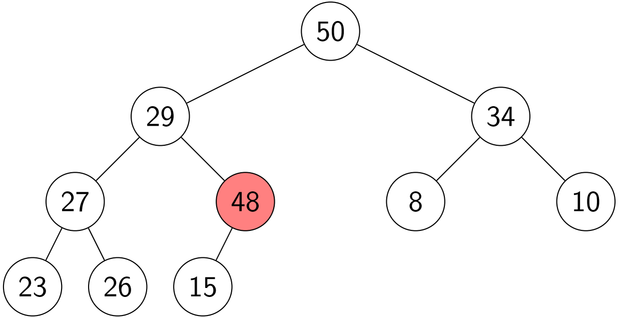

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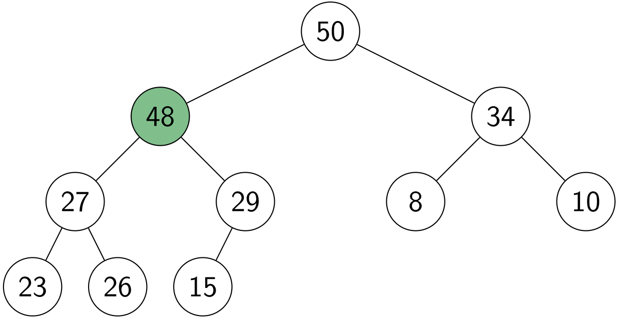

deleteMax in Heaps

- The maximum item of a heap is just the root node.
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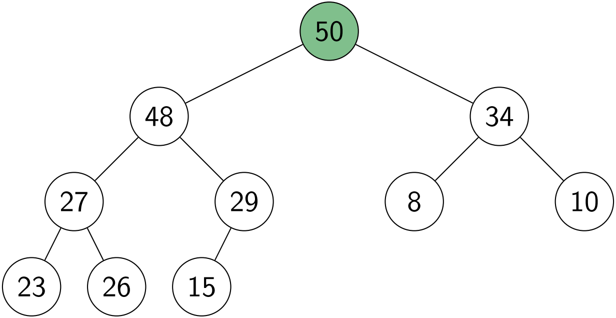
```
fix-down(A, i, n)
A: an array that stores a heap of size n
i: an index corresponding to a node of the heap
    while i is not a leaf do
        j ← left child of i        // Find the child with the larger key
        if (i has right child and A[right child of i].key > A[j].key)
            j ← right child of i
        if A[i].key ≥ A[j].key break
        swap A[j] and A[i]
        i ← j
```

Time: $O(\text{height of heap}) = O(\log n)$.

deleteMax example



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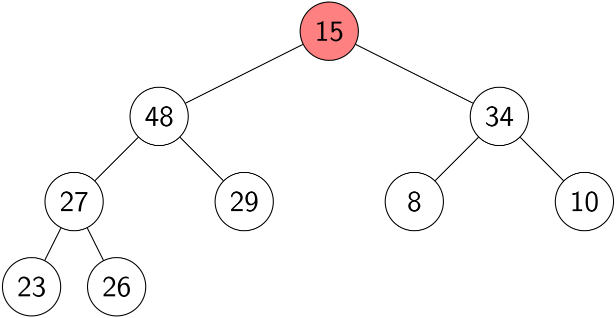

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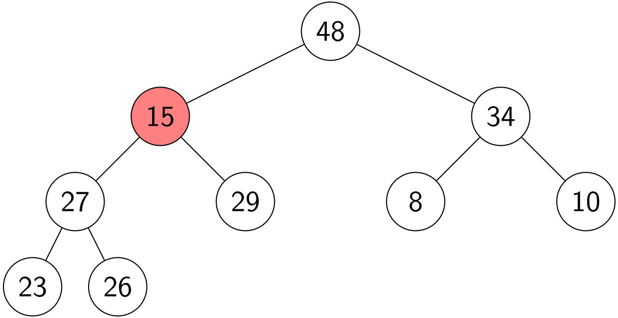

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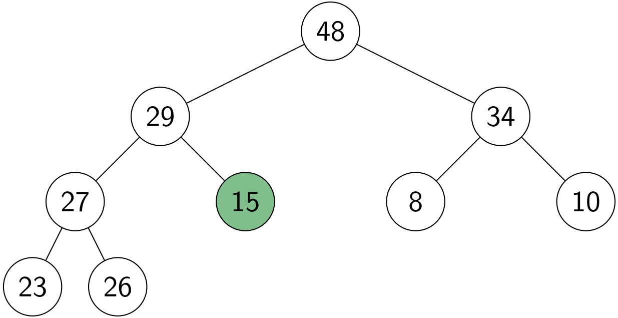

Priority Queue Realization Using Heaps

- Store items in array A and globally keep track of *size*

insert(x)

increase *size*
$\ell \leftarrow$ *last*$()$
$A[\ell] \leftarrow x$
fix-up(A, ℓ)

deleteMax$()$

$\ell \leftarrow$ *last*$()$
swap $A[$*root*$()]$ and $A[\ell]$
decrease *size*
fix-down$(A,$ *root*$(),$ *size*$)$
return $A[\ell]$

insert and *deleteMax*: $O(\log n)$ **time**

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Sorting using heaps

- Recall: Any priority queue can be used to *sort* in time

$$O(\textit{initialization} + n \cdot \textit{insert} + n \cdot \textit{deleteMax})$$

- Using the binary-heaps implementation of PQs, we obtain:

```
PQsortWithHeaps(A)
    initialize H to an empty heap
    for i ← 0 to n − 1 do
        H.insert(A[i])
    for i ← n − 1 down to 0 do
        A[i] ← H.deleteMax()
```

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```

- both operations run in $O(\log n)$ time for heaps

⇝ *PQ-Sort* using heaps takes $O(n \log n)$ time.

- Can improve this with two simple tricks → **Heapsort**
  1. Heaps can be built faster if we know all input in advance.
  2. Can use the same array for input and heap. ⇝ $O(1)$ *auxiliary space!*

Building Heaps with Fix-up

Problem: Given n items all at once (in $A[0 \cdots n-1]$) build a heap containing all of them.

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Solution 1: Start with an empty heap and insert items one at a time:

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simpleHeapBuilding(A)
A: an array
1.    initialize H as an empty heap
2.    for i ← 0 to A.size() − 1 do
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This corresponds to doing *fix-ups*
Worst-case running time: $\Theta(n \log n)$.

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Solution 2: Using *fix-downs* instead:

```
heapify(A)
A: an array
    n ← A.size()
    for i ← parent(last()) downto root() do
        fix-down(A, i, n)
```

Building Heaps with Fix-down

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1. $n \leftarrow A.size()$
2. **for** $i \leftarrow$ *parent*(*last*()) **downto** *root*() **do**
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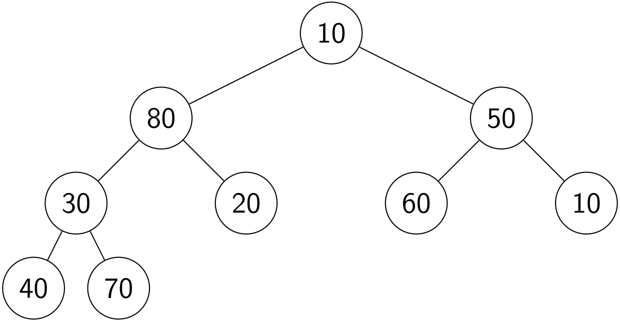
A careful analysis yields a worst-case complexity of $\Theta(n)$.
A heap can be built in linear time.

heapify example



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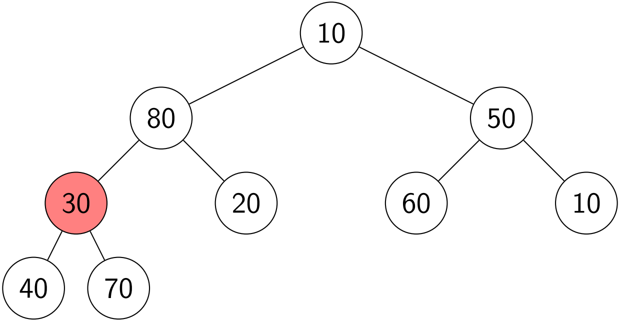

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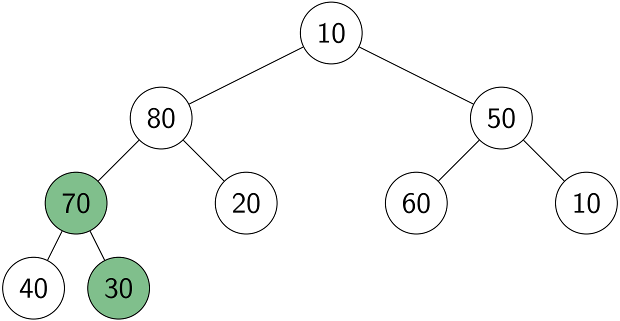

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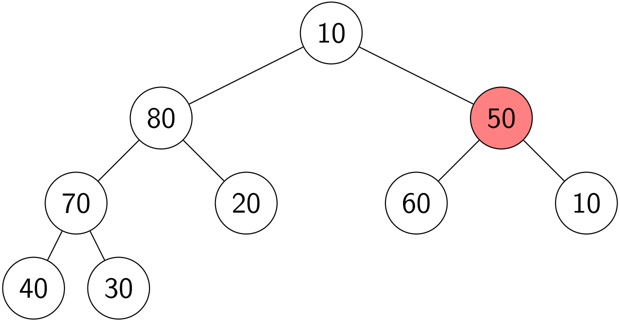

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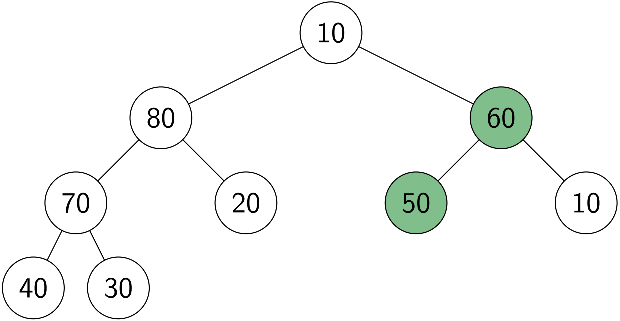

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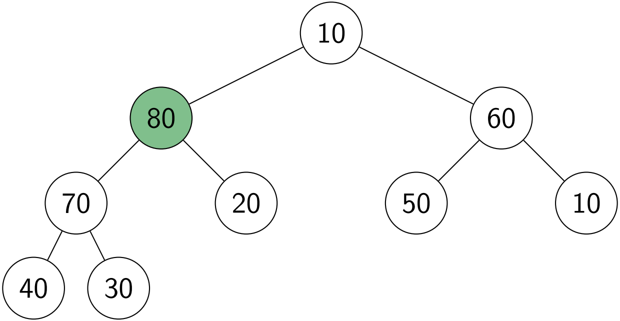

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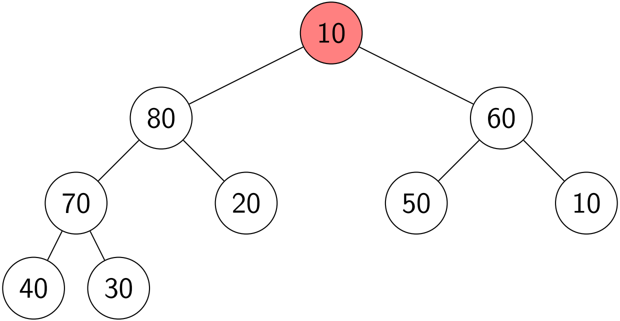

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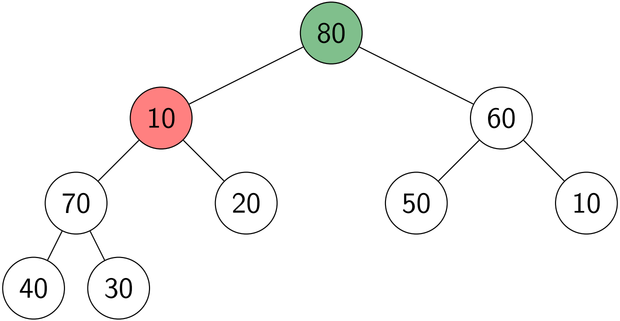

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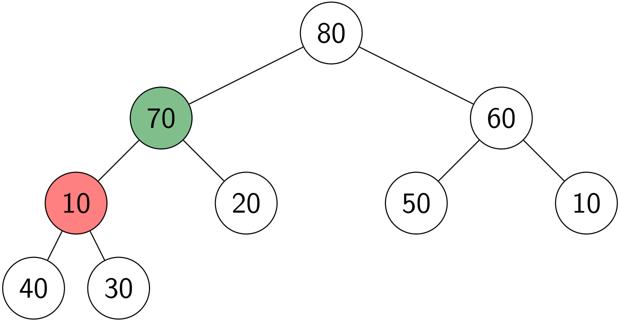

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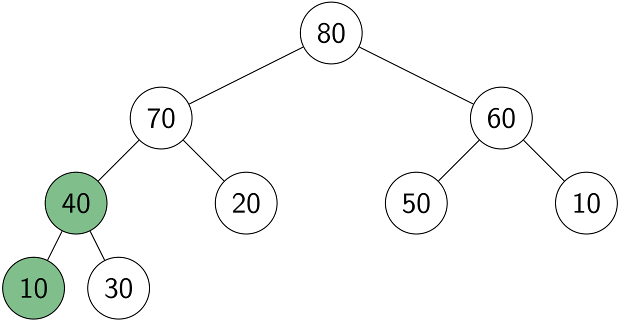


Efficient sorting with heaps

- Idea: *PQ-sort* with heaps.
- $O(1)$ auxiliary space: Use same input-array A for storing heap.

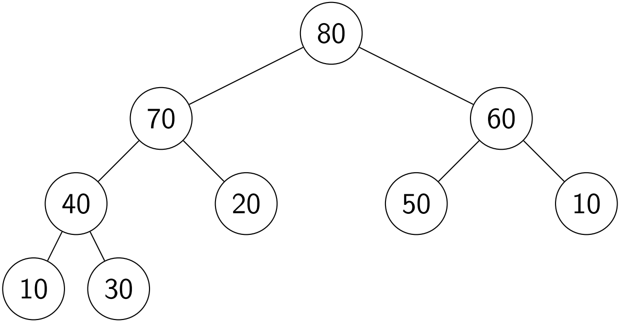
```
HeapSort(A, n)
    // heapify
    n ← A.size()
    for i ← parent(last()) downto 0 do
        fix-down(A, i, n)

    // repeatedly find maximum
    while n > 1
        // 'delete' maximum by moving to end and decreasing n
        swap items at A[root()] and A[last()]
        decrease n
        fix-down(A, root(), n)
```

The for-loop takes $\Theta(n)$ time and the while-loop takes $O(n \log n)$ time.

Heapsort example

Continue with the example from heapify:



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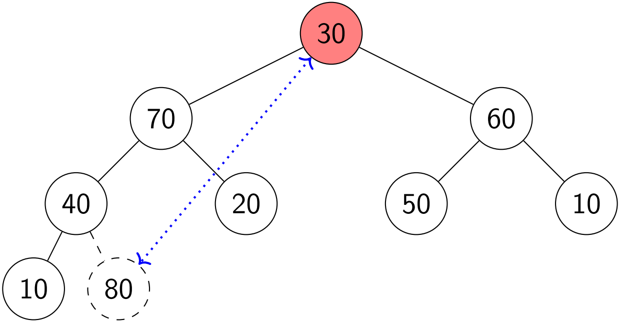

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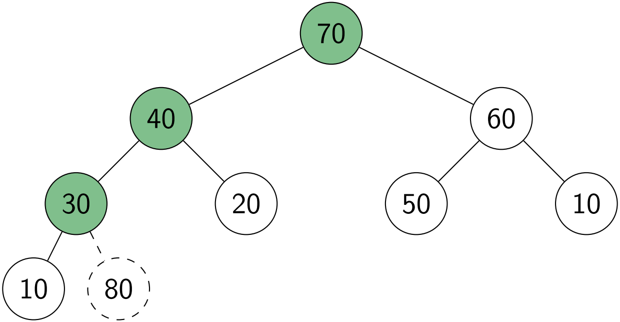

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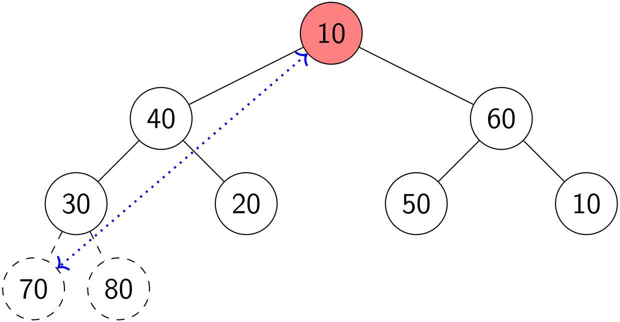

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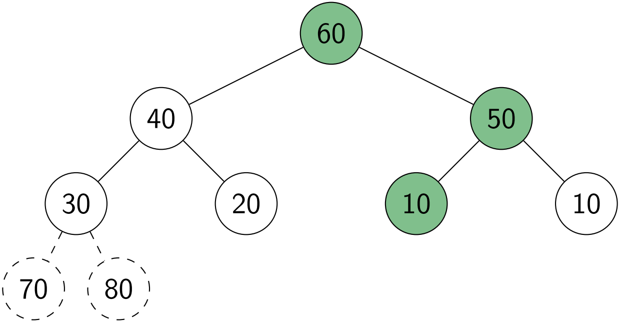

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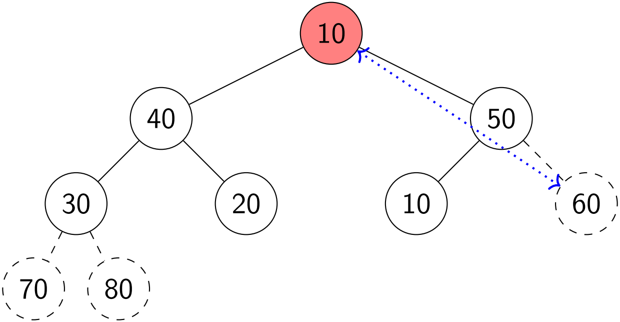

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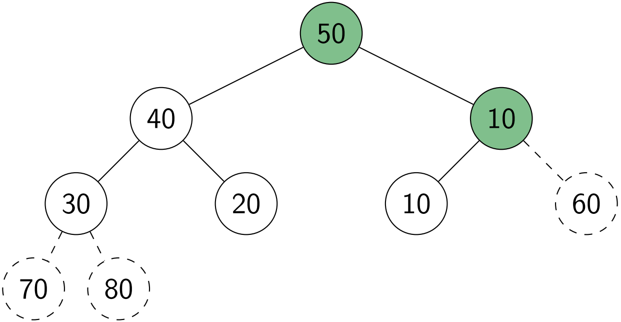

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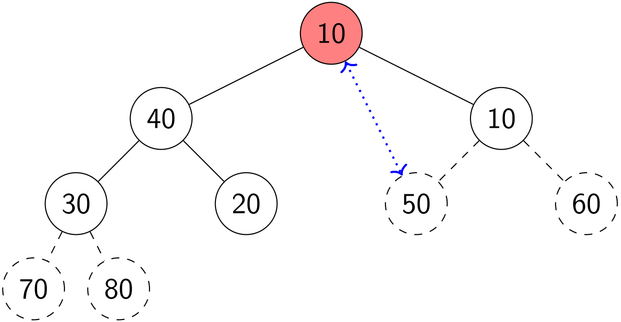

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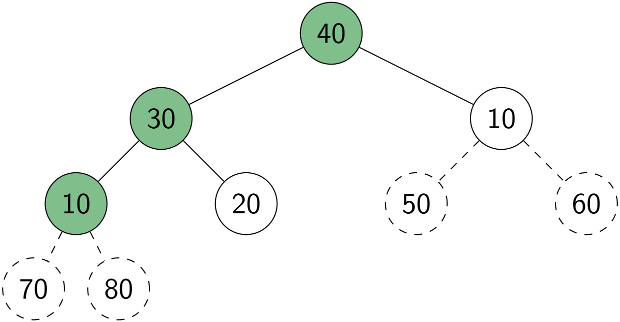

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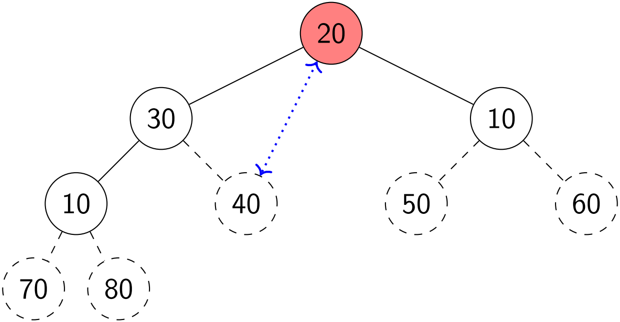
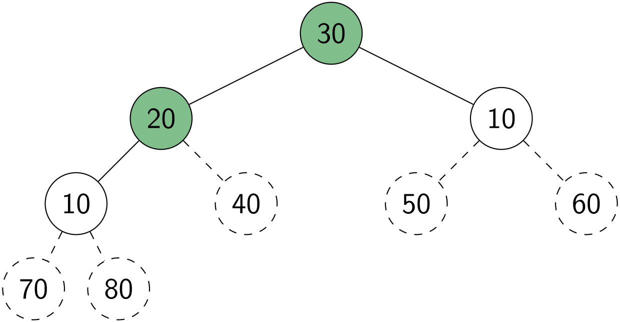


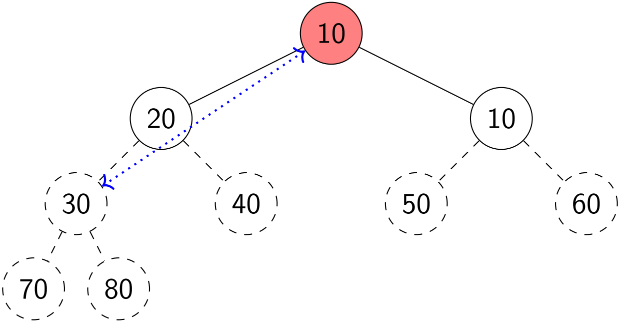
Heapsort example

Continue with the example from heapify:



Heapsort example

Continue with the example from heapify:



Heapsort example

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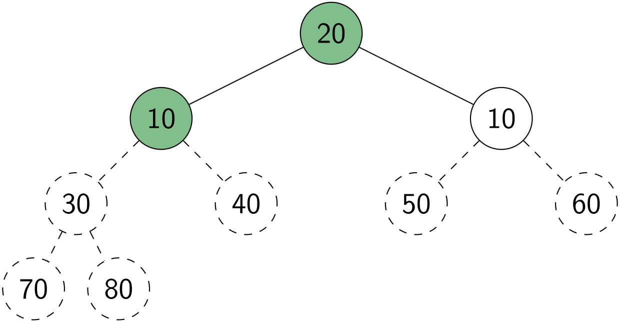

Heapsort example

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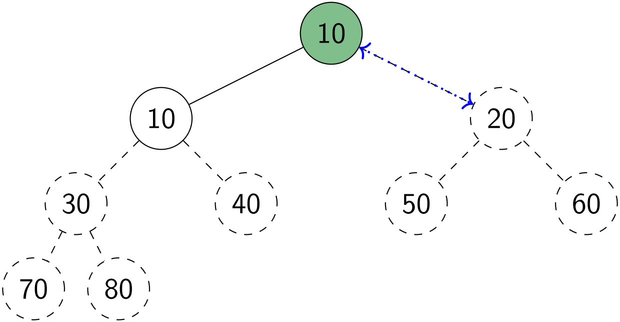

Heapsort example

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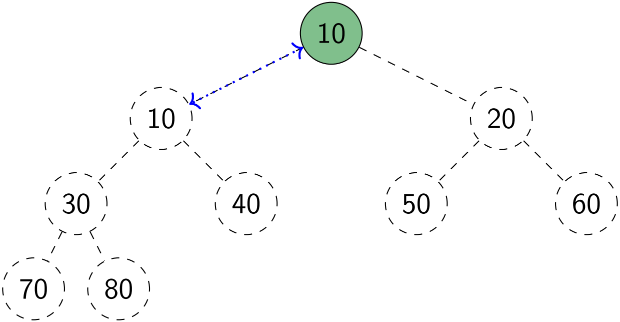

The array (i.e., the heap in level-by-level order) is now in sorted order.

Heap summary

- **Binary heap**: A binary tree that satisfies structural property and heap-order property.
- Heaps are one possible realization of ADT PriorityQueue:
  - *insert* takes time $O(\log n)$
  - *deleteMax* takes time $O(\log n)$
  - Also supports *findMax* in time $O(1)$
- A binary heap can be built in linear time.
- *PQ-sort* with binary heaps leads to a sorting algorithm with $O(n \log n)$ worst-case run-time ($\rightsquigarrow$ *HeapSort*)
- We have seen here the *max-oriented version* of heaps (the maximum priority is at the root).
- There exists a symmetric *min-oriented version* that supports *insert* and *deleteMin* with the same run-times.

Outline

Finding the smallest items

Problem: Find the *kth smallest item* in an array A of n distinct numbers ($k = 0, \ldots, n-1$)

Solution 1: Sort A, then return $A[k]$.
Complexity: $\Theta(n \log n)$.

Solution 2: Make $k+1$ passes through the array, deleting the minimum number each time.
Complexity: $\Theta(kn)$.

Solution 3: Scan the array and maintain the $k+1$ smallest numbers seen so far in a max-heap
Complexity: $\Theta(n \log k)$.

Solution 4: Create a min-heap with *heapify*(A). Call *deleteMin*(A) $k+1$ times.
Complexity: $\Theta(n + k \log n)$.