1 Regular Expression Solutions

1. Naively we can simply enumerate the finite language: $aa|ab|ba|bb$. However, we can also take advantage of concatenation to get $(a|b)(a|b)$.

2. The regular expression for a term is $a|b$, and the regular expression for an operator is $+|−|·|/$. We want to allow one term followed by arbitrarily many operator/term pairs, yielding: $(a|b)((+|−|·|/) (a|b))^∗$.

3. Notice that an even number of 0s can occur in two ways: either there can be an even number on each side of the first 1, or an odd number on each side. First, if we allow exactly one 1 and no 2s, the regular expressions look as follows:
   - $(00)^∗1(00)^∗$
   - $(00)^∗010(00)^∗$

   Next, we want to allow 2s: simply insert $2^∗$ between any two symbols in the above expressions.
   - $2^∗((02^∗02^∗)^∗12^∗(02^∗02^∗)^∗)$
   - $2^∗((02^∗02^∗)^∗2^∗12^∗02^∗(02^∗02^∗)^∗)$

   Next, we want to allow more than a single 1: simply replace every $2^∗$ after the first 1 with a $(1|2)^∗$ to allow arbitrary combinations of 1s and 2s.
   - $2^∗((02^∗02^∗)^∗1(1|2)^∗(0(1|2)^∗0(1|2)^∗)^∗)$
   - $2^∗((02^∗02^∗)^∗02^∗1(1|2)^∗0(1|2)^∗(0(1|2)^∗0(1|2)^∗)^∗)$

   Finally, we need to combine the two expressions. We could simply OR them together, but notice that they only differ by the 1 vs $02^∗1(1|2)^∗0$ in the middle, allowing us to arrive at a slightly shorter final regular expression:

   $$2^∗((02^∗02^∗)^∗(1|02^∗1(1|2)^∗0)(1|2)^∗(0(1|2)^∗0(1|2)^∗)^∗)$$
2 NFA Solutions

- First, we add \{0\} as a starting state (step 1, 2, 3):

Next, we notice that 0 can transition on \(a\) to either 0 or 1, and can transition on \(b\) to 2, so we add those states and transitions (step 4).

Next repeat for state \{0,1\}. Notice that 0 can transition on \(a\) to either 0 or 1 and 1 can transition on \(a\) to 2, so we add a transition from \{0,1\} to \{0,1,2\}. Similarly, \{0,1\} transitions on \(b\) to \{1,2,3\}.

- Next repeat for state \{2\}. On an \(a\) we go to state \{1\}, and on a \(b\) we go to state \{3\}.

- Next repeat for state \{3\}: the only transition is on a \(b\) to state \{1\}.

- Next repeat for state \{1\}: there is a transition on \(a\) to \{2\}, and a transition on \(b\) to \{1,3\}. 

\[
\begin{align*}
\text{start} &\rightarrow \{0\} \\
\{0\} &\rightarrow \{0\} \quad \text{(on } a\text{)} \\
\{0\} &\rightarrow \{0,1\} \quad \text{(on } b\text{)} \\
\{0,1\} &\rightarrow \{0,1,2\} \quad \text{(on } a\text{)} \\
\{0,1,2\} &\rightarrow \{1,2,3\} \quad \text{(on } b\text{)} \\
\{1,2,3\} &\rightarrow \{1\} \quad \text{(on } a\text{)} \\
\{1\} &\rightarrow \{1,3\} \quad \text{(on } b\text{)}
\end{align*}
\]
• Next repeat for state \{1,3\}: there is a transition on \(a\) to \{2\}, and a transition on \(b\) back to \{1,3\}.

• Next repeat for state \{0,1,2\}: there is a transition on \(a\) back to \{0,1,2\}, and a transition on \(b\) to \{1,2,3\}.

• Next repeat for state \{1,2,3\}: there is a transition on \(a\) to \{1,2\}, and a transition on \(b\) to \{1,3\}.

• Next repeat for state \{1,2\}: there is a transition on \(a\) back to \{1,2\}, and a transition on \(b\) to \{1,3\}.
• We’ve finally added transitions to every state. To finish off with step 6, label every state containing an element of $A$ (in this case, every state containing 3) as accepting.
2.1 \( \varepsilon \)-NFA Solutions

First, we add the start state and every state which has
• a non-\( \varepsilon \) transition leading into it. This ends up being all states other than state 1.

• Next, notice that 0 can reach itself on an \( a \) and 2 on a \( b \) (taking an \( \varepsilon \)-transition to state 1 first).

• Next, notice that 2 can reach both itself and 3 on an \( a \) and 0 on a \( b \).

• Next, notice that 3 can reach itself on an \( a \) and everything on a \( b \).

• Finally, both 2 and 3 should be accepting states since they can both reach 2 without any input required.