Exercise 1.
Huth and Ryan, Exercises 2.1, Problem 1

Exercise 2.
Huth and Ryan, Exercises 2.1, Problem 3

Exercise 3.
Huth and Ryan, Exercises 2.2, Problem 1

Exercise 4.
Huth and Ryan, Exercises 2.2, Problem 3

Exercise 5.
For this question, we use the symbols ‘+’, ‘⋅’ and ‘=’ with their normal mathematical meanings: addition, multiplication and equality, respectively. Note that ‘0’, ‘1’, etc., are NOT included as symbols.

(a) Give formulæ for each of the following. Assume that the domain is $\mathbb{N}$, the natural numbers.
   i. “$x$ is the number one.”
   ii. “$x$ is an even number.”
   iii. “Every number is less that its square.”
   iv. “$x$ is composite number.” (Try it first without worrying that neither zero nor one is technically composite. Then check whether your formula handles them correctly, and fix it if not.)

(b) Consider the formula
   $$(\forall y (\forall z (x = y \cdot z \rightarrow (y = x \lor z = x))))$$
   which might be written to mean “$x$ is a prime number”.

   For each of the domains below, explain what it actually means; that is, describe the set of $x$ in the domain that make the formula true. If the set is finite, list its elements; if the set is infinite, describe in one English phrase or sentence the elements of the set.
   i. Domain $\mathbb{N}$, the natural numbers (including zero).
   ii. Domain $\mathbb{Z}$, the integers.
   iii. Domain $\mathbb{Q}$, the rational numbers.

Exercise 6.
On the overheads, you saw the example formula
   $$\left( \forall \varepsilon \left( \varepsilon > 0 \rightarrow \exists \delta \left( \delta > 0 \land \left( \forall y \left| x - y \right| < \delta \rightarrow |f(x) - f(y)| < \varepsilon \right) \right) \right) \right),$$
   which, using the usual interpretation of the real numbers, expresses that function $f$ is continuous at the point $x$. To express that $f$ is continuous everywhere, we can add “$\forall x$ ” in front of that formula.
What about “$f$ is uniformly continuous”? Consider the following formulae – which one(s), if any, express uniform continuity? What do the others express?

(a) $(\forall \varepsilon (\exists \delta (\varepsilon > 0 \rightarrow (\forall x (\forall y (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)))))))$

(b) $(\forall \varepsilon (\exists \delta (\forall x (\forall y (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon)))))))$

(c) $(\forall \varepsilon (\exists \delta (\forall x (\exists \delta (\forall y (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon))))))$

(d) $(\forall \varepsilon (\exists \delta (\exists \delta (\forall x (\exists \delta (\forall y (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon))))))$

Pay particular attention to the quantifiers and their placement in the formulas.

**Exercise 7.**

(a) Let $S$ be a proof system for Propositional Logic. (Recall that a proof system $S$ consists of a set of inference rules.) State carefully what it means for the proof system $S$ to be sound.

(b) Let the Bad proof system for Propositional Logic have one inference rule: for each formula $\phi$,

$$\varphi \vdash_{Bad} (\neg \varphi) .$$

Prove that the Bad proof system is not sound.