Question 1 (12 marks).

[Learning Goals: Write the parse tree for a well-formed formula; Prove properties of a recursively defined concept using structural induction.]

The Reverse Polish notation for Propositional formulae is defined as follows.

- Any Propositional variable (atom) is a formula in Reverse Polish notation.
- If $\alpha$ is a formula in Reverse Polish notation, then $\alpha\neg$ is a formula in Reverse Polish notation.
- If $\alpha$ and $\beta$ are formulae in Reverse Polish notation, then $\alpha\beta\land$, $\alpha\beta\lor$ and $\alpha\beta\rightarrow$ are formulae in Reverse Polish notation.
- Nothing else is a formula in Reverse Polish notation.

Racket uses prefix notation where the operator comes before all the operands. In our notation, we use infix notation, where the operator comes in between the operands. The Reverse Polish notation uses postfix notation where the operators come after all the operands.

For example, the formula $pqr\rightarrow\land$ in Reverse Polish notation would convert to $(p \land (q \rightarrow r))$ in our usual notation.

Throughout this question, let $\gamma$ be the formula $r\neg q\neg p\rightarrow\lor$ in Reverse Polish notation.

(a) Give the well-formed formula of Propositional Logic that corresponds with $\gamma$.

(b) Give the parse tree for $\gamma$. 
(c) Let $\varphi$ be any formula in Reverse Polish notation. Prove by structural induction that any non-empty prefix of $\varphi$ contains more Propositional variables than binary connectives.

Be careful to lay out your induction precisely. You may group similar cases together.
Question 2 (10 marks).

[Learning Goals: Write the parse tree for a well-formed formula. Determine the logical equivalence of formulae.]

Consider the following three well-formed formulæ:

\[(p \to (q \land r)) \quad (p \to q) \land r \quad ((q \lor (\neg p)) \land (r \lor (\neg p))).\]

(a) Write the valuation tree for each of the three formulæ. Assign values for your Propositional variables in alphabetical order.
(b) Write a truth table containing all three formulæ (if you want, you can have separate tables for clarity).

(c) Determine with justification, which of the formulæ, if any, are logically equivalent from 2b.
Question 3 (6 marks).

[Learning Goals: Define tautology, contradiction and satisfiable formula; Prove whether a formula is a tautology, a contradiction or satisfiable.]

Let $\alpha$ be an arbitrary well-formed formula. For each of the following, determine whether it is possible to find a well-formed formula $\beta$ such that the following form a tautology independent of your choice of $\alpha$. That is, either find a $\beta$ (possibly dependent on $\alpha$) such that the following formula will give a tautology or justify that there exists an $\alpha$ such that for any $\beta$, the following formula cannot be a tautology. Do the same with tautology replaced by the word contradiction and then again by the word satisfiable. Justify your answers.

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(a) $(\alpha \land \beta)$

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(b) $(\alpha \rightarrow \beta)$
Question 4 (12 marks).

[Learning Goals: Define tautology, contradiction and satisfiable formula; Prove whether a formula is a tautology, a contradiction or satisfiable using a truth table and/or a valuation tree.]

For each of the Propositional formulas given below, determine with proof whether the formula is a contradiction (i.e. not satisfiable), satisfiable and not a tautology, or a tautology (i.e. a valid formula). Justify your answer by using a truth table, or by using a valuation tree, or by giving a truth valuation, or by writing down your reasoning in English as appropriate.

(a) \((a \land ((b \lor c) \rightarrow \neg a))\)

(b) \(((p \rightarrow q) \lor ((q \rightarrow r) \lor (r \rightarrow p)))\)

(c) \(((b \leftrightarrow ((\neg a) \land c)) \land (b \land a))\)
Question 5 (12 marks).

[Learning Goals: Given a formula, find a valuation that makes it true or false. Evaluate formulæ at a valuation.]

Given the following formulæ:

i) $\alpha = ((a \land ((\neg (b \lor c)) \rightarrow a)) \land ((\neg b) \land (\neg c)))$

ii) $\beta = ((c \rightarrow a) \lor (a \rightarrow (b \lor c)))$

iii) $\gamma = ((c \lor (\neg a)) \leftrightarrow ((\neg (c \lor b)) \land a)) \land c$

answer the following questions.

(a) Find a valuation that makes two of the formulæ true and the other one false.

(b) Find a valuation that makes one of the formulæ true and the other two false.
(c) Is it possible to find a valuation that makes all three formulæ true? Justify your answer.

(d) Is it possible to find a valuation that makes all three formulæ false? Justify your answer.