Question 1 (15 marks).

[Learning Goals: Determine whether a semantic entailment holds by using truth tables, valuation trees, and/or logical identities.]

Prove or disprove each of the following semantic entailment statements. If a semantic entailment statement does not hold give a valuation that demonstrates this. If a semantic entailment statement does hold, either draw a truth table marking the relevant rows or explain why if every formula to the left of the $\models$ evaluates to $T$ under a truth valuation then the formula to the right also evaluates to $T$ under the same truth valuation.

(a) $\{(q \vee r) \rightarrow s\} \models (q \rightarrow s)$

(b) $\{(q \rightarrow r), ((\neg s) \rightarrow (\neg r))\} \models ((q \rightarrow s) \land (s \rightarrow q))$

(c) $\{((\neg r) \rightarrow s), (q \lor (\neg r)), (s \rightarrow (\neg p)), (\neg (\neg p))\} \models q$
(d) \{((\neg r) \lor (\neg p)), ((\neg q) \rightarrow r), ((\neg s) \rightarrow p)\} \models (q \lor s)

(e) \{((\neg r) \lor (\neg p)), ((\neg q) \rightarrow r), ((\neg s) \rightarrow p)\} \models ((\neg q) \lor s)
Question 2 (20 marks).

[Learning Goals: Prove a conclusion from given premises using natural deduction inference rules.]

Give Natural Deduction proofs of the following:

You may only use the following rules: Premise/Reflexivity, ∧i, ∧e, →i, →e, ∨i and ∨e.

Do not use ⊥i/¬e, ¬i, ¬¬e or ⊥e.

[5]

(a) \{ (q \rightarrow r), (r \rightarrow s) \} \vdash (q \rightarrow (r \land s))

[5]

(b) \{ ((q \lor r) \rightarrow s) \} \vdash (q \rightarrow s)
(c) \{((\neg r) \rightarrow s), (q \lor (\neg r)), (s \rightarrow (\neg p)) \} \vdash (q \lor (\neg p))

(d) \{(a \land c), (d \land b)\} \vdash ((a \land b) \land (c \land d))
Question 3 (5 marks).

[Learning Goal: Prove properties of formulas using structural induction.]

For a formula \( \varphi \) we define the set of all subformulas \( S_\varphi \) of \( \varphi \) recursively as follows:

- if \( \varphi \) is \( p \) for a variable \( p \), then \( S_\varphi \) is \( \{p\} \)
- if \( \varphi \) is \( (\neg \alpha) \), then \( S_\varphi \) is \( \{(\neg \alpha)\} \cup S_\alpha \)
- if \( \ast \) is a binary connective and \( \varphi \) is \( (\alpha \ast \beta) \), then \( S_\varphi \) is \( \{(\alpha \ast \beta)\} \cup S_\alpha \cup S_\beta \)

We use \( |\varphi| \) to denote the length of a formula \( \varphi \), and \( |S| \) for number of elements in a set \( S \).

Show by structural induction that the number of subformulas of a well-formed formula is at most its length. That is, show that for every formula \( \varphi \), \( |S_\varphi| \leq |\varphi| \).


**Question 4** (10 marks).


This question considers the following proof written in English:

If there is DNA evidence, then the man was in the room. If he was in the room, he either committed the crime or he witnessed the crime. If he committed the crime, then he ran away and he is hiding. If he witnessed the crime, then he is hiding. There was DNA evidence. Therefore he was in the room. Therefore he committed the crime or he witnessed the crime. In the case that he committed the crime, he ran away and is hiding. In particular he is hiding. In the case that he witnessed the crime, he is hiding. Therefore the man is hiding.

(a) Translate the above argument into propositional logic formulas. Clearly indicate which formulas are the premises of the proof, and which formula is the conclusion of the proof.

(b) Provide a Natural Deduction proof for the above argument. Your Natural Deduction proof must follow the same structure as the English version.