Due Monday, June 4, by 4:00pm, to Crowdmark.

All submitted work must be the student’s own.

**Question 1** (9 marks).

_Learning Goal: Prove a conclusion from given premises using Natural Deduction inference rules._

Prove each of the following using natural deduction.

For full credit, use ONLY the basic rules of the Natural Deduction system (\(\land i, \land e, \lor i, \lor e, \rightarrow i, \rightarrow e, \neg i, \neg e \) (or \(\bot i\)), \(\bot e\), and \(\neg \neg e\)).

You may receive partial credit if you use one of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or \(\neg \neg i\).

(a) \([\neg (r \lor q), (s \rightarrow (p \lor q)), (p \rightarrow r)] \vdash_{ND} (\neg s)\).

(b) \([\neg p] \vdash_{ND} (p \rightarrow q)\).
(c) \{r\} \vdash_{ND} ((q \land s) \lor (s \rightarrow r))$.

(d) \{\lnot s\} \vdash_{ND} ((q \land s) \lor (s \rightarrow r))$. 
Question 2 (8 marks).
[Learning Goal: Prove a conclusion from given premises using Natural Deduction inference rules.]

Prove each of the following using natural deduction.
You may use any of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or $\neg\neg$-i, without loss of credit.

(a) $\emptyset \vdash_{ND} (((p \rightarrow q) \rightarrow p) \rightarrow p)$.

Hint: You may want to use your solution to Q1b in this proof.
(b) \((q \lor r) \vdash_{ND} ((q \land s) \lor (s \rightarrow r))\).

Hint: We highly recommend using the derived rule LEM on \(s\) in your proof. You may want to use your solutions to Q1c and Q1d in this proof.
Question 3 (7 marks).

[Learning Goals: Translate English sentences into propositions. Prove a conclusion from given premises using natural deduction inference rules.]

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. A knight invited a newcomer to the island. The newcomer met two inhabitants: Zoey and Mel. Zoey tells the newcomer that Mel is a knave. Mel says, “Zoey and I are both knights.”

Using the power of logic, the newcomer concluded that Zoey is a knight and Mel is a knave.

Remark: There are no natural deduction inference rules for $\leftrightarrow$. If, when answering this question, you need to write a formula $(\alpha \leftrightarrow \beta)$, write $((\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha))$ instead.

(a) Translate the above argument into propositional formulas. Clearly indicate which formulas are the premises, and which formula is the conclusion.

Hint: A sentence “person x says y” is equivalent to “person x is a knight if and only if y is true”.


(b) Provide a Natural Deduction proof for the above argument.
You may use any of the derived rules MT (*modus tollens*), PBC (*proof by contradiction*), LEM (*law of excluded middle*), or $\neg\neg i$, without loss of credit.
Question 4 (3 marks).
[Learning Goal: Prove the soundness of an inference rule.]

Consider the following inference rule

\[ \frac{(\alpha \rightarrow \beta) \quad (\beta \rightarrow \alpha) \quad (\alpha \lor \beta)}{(\alpha \land \beta)} \]

where \( \alpha \) and \( \beta \) are any well-formed Propositional formulæ.

Prove that the \( \land i^* \) inference rule is sound, i.e. prove that the following semantic entailment holds.

\( \{(\alpha \rightarrow \beta), (\beta \rightarrow \alpha), (\alpha \lor \beta)\} \models (\alpha \land \beta) \).

You must use the definition of semantic entailment to write your proof. Do not use any other technique such as truth tables, valuation trees, logical equivalence, soundness, or completeness.
Question 5 (6 marks).

[Learning Goal: Prove/disprove a semantic entailment from a premise using soundness and completeness.]

Let $\alpha$ and $\beta$ be well-formed Propositional formulae.

(a) Prove or disprove the statement: If $\{\alpha\} \models \beta$, then $\emptyset \vdash (\alpha \rightarrow \beta)$. 

(b) Prove or disprove the statement: If $\emptyset \vdash (\beta \rightarrow \alpha)$, then $\{\alpha\} \models \beta$. 
