Due Friday, Oct. 13, by 4:00pm, to Crowdmark.
All submitted work must be the student’s own.

Question 1 (9 marks).
[Learning Goal: Prove a conclusion from given premises using Natural Deduction inference rules.]

Give proofs in Natural Deduction showing each of the following. For full credit, use ONLY the basic rules of the Natural Deduction system (Huth and Ryan, top of p. 27, or Syntax and Semantics..., p. 4, from the Web page) on the first two. You may receive partial credit if you use one of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or ¬¬i.

(a) \{ (\neg (r \lor q)), (s \rightarrow (p \lor q)), (p \rightarrow r) \} \vdash (\neg s).
[3]  
(b) \{q \lor r\} \vdash ((q \land s) \lor (s \rightarrow r)).
(c) $\emptyset \vdash ((p \rightarrow q) \rightarrow p) \rightarrow p)$. For part (c), you may use any of the derived rules listed at the start of the question without loss of credit.
Question 2 (9 marks).

[Learning Goals: Prove semantic entailment using truth tables and/or valuation trees. Prove a conclusion from given premises using Natural Deduction inference rules.]

For each of the following, determine whether or not the indicated Natural Deduction proof exists. If a proof exists, then give one. You may use any derived rules in your proof without loss of credit. If no proof exists, then carefully explain why not.

(a) \( \{(p \to r), (r \to p), ((p \land q) \to r)\} \vdash ((p \lor q) \to r) \).

(b) \( \{(p \to r), (r \lor p), ((p \land q) \to r)\} \vdash ((p \lor q) \to r) \).
(c) \{((\neg p) \rightarrow (\neg r)), (\neg (s \lor p)), (q \rightarrow (r \lor s))\} \vdash (\neg q).
Question 3 (4 marks).
[Learning Goal: Prove properties of a recursively defined concept using induction.]

Remark on Notation: In this question, we use the associativity of $\wedge$ and $\vee$ to omit parentheses where doing so does not create ambiguity.

Let $p_0, p_1, p_2, \ldots$ be Propositional variables. Let $n$ be any natural number. Prove, using induction on $n$, the following Generalized DeMorgan Law, for any natural number $n$:

$$(\neg(p_n \wedge \cdots \wedge p_0)) \equiv (\neg p_n) \vee \cdots \vee (\neg p_0).$$
Question 4 (6 marks).
[Learning Goal: Prove the soundness of Resolution.]

A proof system which has been taught in CS 245 in the past is called Resolution. The Resolution proof system has one inference rule, and its simplest form is

\[
\frac{(\alpha \lor p) \quad ((\neg p) \lor \beta)}{(\alpha \lor \beta)}
\]

for any Propositional variable \( p \) and any well-formed Propositional formulæ \( \alpha \) and \( \beta \). Prove that this inference rule is sound, i.e. prove the semantic entailment

\[
\{ (\alpha \lor p), ((\neg p) \lor \beta) \} \vDash (\alpha \lor \beta).
\]

Note that since we do not know which Propositional variables are used in \( \alpha \) and \( \beta \), it is not possible to write down a truth table or a valuation tree for \( \alpha \) or for \( \beta \).
Question 5 (8 marks).

[Learning Goal: Prove/disprove a given conclusion using soundness and completeness.]

Let $\alpha, \beta$ be well-formed Propositional formulæ. Recall that $\emptyset$ denotes the empty set, which in this question is the set of no formulæ.

(a) If $\{\alpha\} \not\models \beta$, does it follow that $\emptyset \not\vdash (\alpha \rightarrow \beta)$? If it does follow, then give a proof. If it does not follow, then give a choice of $\alpha, \beta$ such that $\{\alpha\} \not\models \beta$ but $\emptyset \vdash (\alpha \rightarrow \beta)$ and explain clearly why your choice is correct.

(b) If $\emptyset \not\vdash (\alpha \rightarrow \beta)$, does it follow that $\{\alpha\} \not\models \beta$? If it does follow, then give a proof. If it does not follow, then give a choice of $\alpha, \beta$ such that $\emptyset \not\vdash (\alpha \rightarrow \beta)$ but $\{\alpha\} \models \beta$ and explain clearly why your choice is correct.