Due Wednesday, Oct.24, by 4pm, to Crowdmark.

All submitted work must be the student's own.

Question 1 (12 marks).

Consider the social networking service FOLogic (it is similar to Twitter). Every user of the service is identified by a unique username and users have the ability to follow users of their choosing (including following themselves).

Let this service be modelled under the following specifications:

Domain: "All the usernames for the users of FOLogic".

Constant Symbols with their interpretations:

$$c : \text{TheCarmen}\ ,\quad j : \text{JonathanB}\ ,\quad s : \text{SMcIntyre}$$

Predicate Symbols with their interpretations:

$$F(x, y) : \text{“x follows y.”}$$

$$S(x, y) : \text{“x is the same user as y.”}$$

Function Symbols with their interpretations:

$$p(x) : \text{“x's most popular follower.”}$$

Where popularity is determined by the number of followers x has.
Ties are broken by the alphabetical order. If x has no followers, return x.

Logical Connectives:

$$\neg,\ \ \wedge,\ \ \vee,\ \ \rightarrow$$

Quantifiers:

$$\forall,\ \exists$$

Using only the above specifications, translate the following sentences into well-formed formulas of Predicate logic. Please ensure your translations follow from the given statements.

[2] (a) "No user follows themselves."

[2] (b) "Someone follows TheCarmen's most popular follower."

[2] (c) "Nobody follows everybody, but everyone follows someone."

[2] (d) "At most one user follows SMcIntyre."

[2] (e) "There are at least two distinct users who follow users with JonathanB as their most popular follower."

[2] (f) "If someone's most popular follower is themselves, then no other user follows them."

Question 2 (8 marks).

In databases, values for relations can be entered in tables. FOLogic is still in development and currently has only three users: SMcIntyre, JonathanB, and TheCarmen.
The *follows* relation, $F(x, y)$, can be seen in the following table:

x	y
JonathanB	TheCarmen
SMcIntyre	JonathanB
SMcIntyre	TheCarmen
TheCarmen	JonathanB

Let the specifications given in **Question 1** and the table above form the interpretation $\mathcal{I}$.
Using only this information, answer the following questions:

[2] (a) Define both $F^{\mathcal{I}}$ and $p^{\mathcal{I}}$.

[2] (b) Is it the case that $\mathcal{I} \vDash \left(\forall x \left(\exists y \left(F(x, y) \rightarrow (\neg S(x, y))\right)\right)\right)$? Explain.

[2] (c) Define an environment E such that $\mathcal{I} \models_E \big(\forall x\, (\neg F(x, y))\big)$
Explain why this is a correct choice for E.

[2] (d) Define an environment E such that $\mathcal{I} \not\models_E \big(\exists x\, (\forall y (F(x, p(z)) \vee F(y, p(z))))\big)$
Explain why this is a correct choice for E.

Question 3 (8 marks).

Let F be a binary predicate symbol. Let p be a unary function symbol.
Let x, y, and z be variable symbols.

Determine which of the following entailments (logical consequences) hold.
If the entailment holds, then show it using the definition of satisfaction of a predicate formula.
Else, give an interpretation and environment where it doesn't hold. Use the domain $\{-1, 0, 1\}$.

[2] (a) $\big(\forall x\,(\exists y\; F(x, y))\big) \;\vDash\; \big(\exists x\,(\forall y\; F(x, y))\big)$

[2] (b) $\big(\exists y\,(\forall x\; F(x, y))\big) \;\vDash\; \big(\forall x\,(\exists y\; F(x, y))\big)$

[2] (c) $\vDash \ \big(\exists x\,(F(x,y) \to F(z,y))\big)$

[2] (d) $\vDash \ \big(\forall x\,(F(x,p(x)) \to F(p(x),x))\big)$

Question 4 (10 marks).

Let f be a unary function symbol, g be a binary function symbol, P and Q be binary predicate symbols, and v, w, x, y, z be variable symbols.
Given the following well-formed predicate formula,

$$\alpha \stackrel{\mathsf{def}}{=} \Big(\forall x\, \big((\exists y\ P(x, y)) \wedge Q(f(z), y)\big)\Big)$$

answer the following questions.

[1] (a) Give the parse tree corresponding to α.

[1] (b) Which occurrences of the variable(s) are **free** in α? You can explicitly state the occurrences by labeling the variables at the leaf nodes of the parse tree numerically from left to right.

[1] (c) Which occurrences of the variable(s) are **bound** in α? You can explicitly state the occurrences by referencing labels on the leaf nodes of the parse tree numerically from left to right.

[1] (d) Determine the scopes of $\forall x$ and $\exists y$ in α.

[2] (e) State $\alpha[g(v,\ f(w))/y]$.

[2] (f) State $\alpha[g(x,\ z)/z]$.

[2] (g) State $\alpha[g(x,\ z)/z][g(v, f(w))/z]$.

Question 5 (6 marks).

A substitution s is of the form $[t_1/x_1] ... [t_n/x_n]$ for terms t_1 to t_n and variables x_1 to x_n. If two substitutions s_1 and s_2 agree on variable x, then $xs_1 = xs_2$.

Let t be a term. Let s_1 and s_2 be substitutions. Prove the following statement:

> If s_1 and s_2 agree on all variables in t, then $ts_1 = ts_2$.

Hint: Use structural induction on t.