Due Wednesday, June 20, by 4:00pm, to Crowdmark.
All submitted work must be the student’s own.

**Question 1** (14 marks).

[Learning Goals: Translate English sentences into Predicate (or first-order) logic statements. Identifying constants and predicates.]

Consider the following specifications:

**Domain**

"all the people in the world”.

**Constant Symbols**

\( a \): Alice, \( c \): Carmen, \( \ell \): Collin

**Function Symbols with their interpretations**

\( f(x) \): returns the father of \( x \).
\( m(x) \): returns the mother of \( x \).

**Predicate Symbols with their interpretations**

\( = \): with the usual meaning.
\( \neq \): with the usual meaning.
\( Y(x, y) \): “\( x \) is younger than \( y \)”.

**Logical Connectives**

\( \rightarrow, \neg, \lor, \land \)

**Quantifiers**

\( \forall, \exists \)

Using only the above specifications, translate the following sentences into well-formed formulæ of Predicate logic. Please ensure your translations follow from the given statements. You may assume no two people are the same age (we count nanosecond differences!)
(a) Alice is younger than everyone’s paternal grandmother.

(b) No one is older than themself.

(c) Someone is older than Collin’s mother.

(d) Nobody is younger than everybody.

(e) Everybody is younger than at least one of Carmen’s maternal grandparents.

(f) At most one person is younger than Alice’s father.

(g) There are at least two distinct people that are older than Carmen’s father.
Question 2 (13 marks).

[Learning Goals: Translate English sentences into Predicate (or first-order) logic statements. Identifying constants and predicates.]

A ring is a set $R$ equipped with two binary functions $+$ and $\cdot$, for addition and multiplication respectively, that satisfies the following properties:

- Addition is associative.
- Addition is commutative.
- Addition has an identity element, that is, an element $0$ such that $x + 0 = x$ for any element $x$ in $R$.
- Each element has an additive inverse, that is, for any element of $R$, say $x$, there exists an element $y$ in $R$ satisfying the addition of $x$ and $y$ is $0$.
- Multiplication is associative.
- Multiplication has an identity element, that is, an element $1$ such that $x \cdot 1 = x$ for any element $x$ in $R$.
- The distributive property holds, that is, for any elements $a$, $b$ and $c$ inside $R$, we have
  \[ a \cdot (b + c) = (a \cdot b) + (a \cdot c) \]
  \[ (b + c) \cdot a = (b \cdot a) + (c \cdot a) \]

In this problem, you will write down these conditions using Predicate (First Order) Logic. Use the binary function symbols $A(x, y)$ for addition and $M(x, y)$ for multiplication and use the binary predicate symbol $E(x, y)$ for equality of $x$ and $y$, For all parts, your formulas must be well-formed formulas in Predicate logic.

(a) What is the domain for this problem?

(b) What, if any, are the constant symbols for this problem? (Write “none” if none).

(c) We say that a ring is commutative if in addition to the above, multiplication is also commutative. Write this condition using Predicate Logic.

(d) We say that a commutative ring is a field if in addition to the above, we have that for every non-zero element $x$ of $R$ has a multiplicative inverse, that is, some element $y$ satisfying $x \cdot y = 1$. Write this condition using Predicate Logic.
Using the function symbols $A(x, y)$ for addition and $M(x, y)$ for multiplication and using the predicate symbol $E(x, y)$ for equality of $x$ and $y$, write down each of the conditions for a set $R$ to be a ring in the same order as presented on the previous page using Predicate Logic.
Question 3 (10 marks).

[Learning Goals: Construct a parse tree for a Predicate formula. Determine free and bound variables of a given expression. Determine the scopes of quantifiers in a formula.]

Let $f$ be a unary function symbol, $P$ and $Q$ be binary predicate symbols, $R$ be a ternary predicate symbol and assume that the following has no constants (in other words, $x, y, z, a, b$ are all variables). Given the following well-formed predicate formula:

$$
\alpha \overset{\text{def}}{=} (\forall x (\exists y ((P(x, y) \lor P(y, x)) \land Q(f(a), y)) \rightarrow (\exists z (Q(x, y) \land (P(f(x), y) \rightarrow R(a, b, z)))))
$$

answer the following questions.

(a) Give the parse tree corresponding to $\alpha$.
(b) Which occurrences of the variable(s) are **free** in $\alpha$? You can explicitly state the occurrences by labeling the variables at the leaf nodes of the parse tree in (3a) numerically from left to right (e.g., $v(1)$ or $v_1$ for $v$ being leftmost leaf node of the tree).

(c) Which occurrences of the variable(s) are **bound** in $\alpha$? You can explicitly state the occurrences by referencing labels on the leaf nodes of the parse tree in (3a) from left to right.

(d) Determine the scopes of $\forall x$ and $\exists z$ in $\alpha$. 
(e) State $\alpha[g(x, z)/a]$.

(f) If $\beta \equiv \alpha[g(x, z)/a]$, state $\beta[h(x, a)/b]$. 
Question 4 (14 marks).

[Learning Goals: Selecting an interpretation that proves or disproves validity of Predicate logic formulas.]

You are given the following well-formed Predicate formula

\[
\gamma \overset{\text{def}}{=} \left( \exists x \left( \forall y \ P(x, f(y, z)) \right) \right)
\]

and

\[
D \overset{\text{def}}{=} \mathbb{Z}
\]

where \( P \) is a Predicate symbol (of arity two), \( f \) is a function symbol (of arity two), \( x, y, z \) are variables and \( D \) is the domain of interpretations \( I_1 \) and \( I_2 \).

(a) Suppose that \( I_1 \) is an interpretation such that \( f^{I_1} \) maps \((y, z)\) to \( y \cdot z \), the product of \( y \) and \( z \) in \( \mathbb{Z} \). Complete the interpretation of \( I_1 \) and give the definition of an environment \( E_1 \) such that

\[
I_1 \models_{E_1} \gamma \text{ or equivalently } \gamma^{(I_1, E_1)} = T.
\]

Clearly explain why your choices are correct.

(b) Suppose that \( I_2 \) is an interpretation such that

\[
P^{I_2} = \{ (a, b) \in D^2 : |a - b| = 1 \}
\]

with the usual notions of absolute value and subtraction over the integers. Complete the interpretation of \( I_2 \) and of an environment \( E_2 \) such that

\[
I_2 \not\models_{E_2} \gamma \text{ or } \gamma^{(I_2, E_2)} = F.
\]

Clearly explain why your choices are correct.
Remark: Parts c) and d) below show that the given formula is satisfiable and not valid.

Now, you are given the same well-formed Predicate formula

$$\gamma \overset{\text{def}}{=} \exists x \left( \forall y \, P(x, f(y, z)) \right)$$

but we change the domain to

$$D \overset{\text{def}}{=} \{-1, 0, 1\}$$

where $P$ is a Predicate symbol (of arity two), $f$ is a function symbol (of arity two) and $D$ is the domain of interpretations $J_1$ and $J_2$.

(c) Give an interpretation $J_1$ and an environment $G_1$ such that

$$J_1 \Vdash_{G_1} \gamma$$

or $\gamma(J_1, G_1) = T$.

Clearly explain why your choices are correct.

(d) Give an interpretation $J_2$ and an environment $G_2$ such that

$$J_2 \not\Vdash_{G_2} \gamma$$

or $\gamma(J_2, G_2) = F$.

Clearly explain why your choices are correct.
Question 5 (10 marks).

Let $\mathcal{L}$ be a language with variables $x$ and $y$, and Predicate symbol $P$ (of arity 2). Determine which of the following formulas are valid. If a formula is not valid, create an interpretation and environment that makes the formula false with the domain $\mathbb{N}$. If a formula is valid, justify why.

(a) \[ ((\forall x (\exists y P(x, y))) \rightarrow (\exists y (\forall x P(x, y)))) \]

(b) \[ ((\forall x (\exists y P(x, y))) \rightarrow (\exists x (\forall y P(x, y)))) \]
(c) \((∃y (∀x P(x, y))) → (∀x (∃y P(x, y)))\)

(d) \((∃y (∀x P(x, y))) → (∀y (∃x P(x, y)))\)

(e) \((∀x (∃y P(x, y))) → (∀y (∃x P(x, y)))\)
**Question 6** (8 marks).

Let $\varphi$ be a well-formed Predicate formula. Prove by structural induction on $\varphi$ that every pair of distinct predicate symbols in $\varphi$ has a binary connective between them.