Question 1 (12 marks).

Consider the social networking service FOLoLogic (it is similar to Twitter). Every user of the service is identified by a unique username and users have the ability to follow users of their choosing (including following themselves).

Let this service be modelled under the following specifications:

Domain: “All the usernames for the users of FOLoLogic”.

Constant Symbols with their interpretations:

\[ c : \text{TheCarmen}, \quad j : \text{JonathanB}, \quad s : \text{SMcIntyre} \]

Predicate Symbols with their interpretations:

\[ F(x, y) : \text{“x follows y.”} \]
\[ S(x, y) : \text{“x is the same user as y.”} \]

Function Symbols with their interpretations:

\[ p(x) : \text{“x’s most popular follower.”} \]

*Where popularity is determined by the number of followers x has. Ties are broken by the alphabetical order. If x has no followers, return x.*

Logical Connectives:

\[ \neg, \quad \land, \quad \lor, \quad \rightarrow \]

Quantifiers:

\[ \forall, \quad \exists \]

Using only the above specifications, translate the following sentences into well-formed formulas of Predicate logic. Please ensure your translations follow from the given statements.
(a) “No user follows themselves.”

(b) “Someone follows TheCarmen’s most popular follower.”

(c) “Nobody follows everybody, but everyone follows someone.”

(d) “At most one user follows SMcIntyre.”

(e) “There are at least two distinct users who follow users with JonathanB as their most popular follower.”

(f) “If someone’s most popular follower is themselves, then no other user follows them.”
Question 2 (8 marks).

In databases, values for relations can be entered in tables. FOLogic is still in development and currently has only three users: SMcIntyre, JonathanB, and TheCarmen.

The follows relation, $F(x,y)$, can be seen in the following table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>JonathanB</td>
<td>TheCarmen</td>
</tr>
<tr>
<td>SMcIntyre</td>
<td>JonathanB</td>
</tr>
<tr>
<td>SMcIntyre</td>
<td>TheCarmen</td>
</tr>
<tr>
<td>TheCarmen</td>
<td>JonathanB</td>
</tr>
</tbody>
</table>

Let the specifications given in Question 1 and the table above form the interpretation $\mathcal{I}$. Using only this information, answer the following questions:

(a) Define both $F^\mathcal{I}$ and $p^\mathcal{I}$.

(b) Is it the case that $\mathcal{I} \models \left( \forall x \left( \exists y \left( F(x,y) \rightarrow \neg S(x,y) \right) \right) \right)$? Explain.
(c) Define an environment $E$ such that $\mathcal{J} \models_E (\forall x (\neg F(x, y)))$
Explain why this is a correct choice for $E$.

(d) Define an environment $E$ such that $\mathcal{J} \not\models_E (\exists x (\forall y (F(x, p(z)) \lor F(y, p(z)))))$
Explain why this is a correct choice for $E$. 
**Question 3** (8 marks).

Let $F$ be a binary predicate symbol. Let $p$ be a unary function symbol. Let $x, y,$ and $z$ be variable symbols.

Determine which of the following entailments (logical consequences) hold. If the entailment holds, then show it using the definition of satisfaction of a predicate formula. Else, give an interpretation and environment where it doesn’t hold. Use the domain $\{-1, 0, 1\}$.

1. $(\forall x (\exists y F(x, y))) \models (\exists x (\forall y F(x, y)))$
2. $(\exists y (\forall x F(x, y))) \models (\forall x (\exists y F(x, y)))$
[2]  (c) \models (\exists x \ (F(x, y) \rightarrow F(z, y)))

[2]  (d) \models (\forall x \ (F(x, p(x)) \rightarrow F(p(x), x))))
Question 4 (10 marks).

Let $f$ be a unary function symbol, $g$ be a binary function symbol, $P$ and $Q$ be binary predicate symbols, and $v, w, x, y, z$ be variable symbols.

Given the following well-formed predicate formula,

$$\alpha \overset{\text{def}}{=} (\forall x ((\exists y P(x, y)) \land Q(f(z), y)))$$

answer the following questions.

(a) Give the parse tree corresponding to $\alpha$. [1]
(b) Which occurrences of the variable(s) are **free** in $\alpha$? You can explicitly state the occurrences by labeling the variables at the leaf nodes of the parse tree numerically from left to right.

(c) Which occurrences of the variable(s) are **bound** in $\alpha$? You can explicitly state the occurrences by referencing labels on the leaf nodes of the parse tree numerically from left to right.

(d) Determine the scopes of $\forall x$ and $\exists y$ in $\alpha$. 
(e) State $\alpha[g(v, f(w))/y]$. 

(f) State $\alpha[g(x, z)/z]$. 

(g) State $\alpha[g(x, z)/z][g(v, f(w))/z]$. 
Question 5 (6 marks).

A substitution $s$ is of the form $[t_1/x_1] \ldots [t_n/x_n]$ for terms $t_1$ to $t_n$ and variables $x_1$ to $x_n$. If two substitutions $s_1$ and $s_2$ agree on variable $x$, then $xs_1 = xs_2$.

Let $t$ be a term. Let $s_1$ and $s_2$ be substitutions. Prove the following statement:

If $s_1$ and $s_2$ agree on all variables in $t$, then $ts_1 = ts_2$.

*Hint: Use structural induction on $t$.*