Due Wednesday, June 27, by 4:00pm, to Crowdmark.

All submitted work must be the student’s own.

Question 1 (12 marks).

[Learning Goal: Prove semantic entailment by using a direct proof or a proof by contradiction.]

Prove each of the given semantic entailments.

(a) \[ \{ (\forall x (\exists y (P(x) \lor Q(y))) \} \models (\exists y (\forall x (P(x) \lor Q(y))) \} \]
(b) \[ \left\{ \forall x \left( \forall y (P(y) \rightarrow Q(x)) \right) \right\} \models \left( \exists y \left( P(y) \rightarrow (\forall x Q(x)) \right) \right) \]
Question 2 (18 marks).

[Learning Goal: Disprove semantic entailment by giving a counterexample.]

Consider the following three Predicate logic sentences that state that (in all interpretations) a binary predicate symbol $P$ is reflexive, symmetric, and transitive:

$$\varphi_r = (\forall x P(x, x))$$
$$\varphi_s = (\forall x (\forall y (P(x, y) \rightarrow P(y, x))))$$
$$\varphi_t = (\forall x (\forall y (\forall z (P(x, y) \rightarrow (P(y, z) \rightarrow P(x, z)))))$$

(a) Show that $\{\varphi_s, \varphi_r\} \not\models \varphi_t$.

(b) Show that $\{\varphi_s, \varphi_t\} \not\models \varphi_r$. 

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(c) Show that \( \{ \varphi_r, \varphi_t \} \not\models \varphi_s \).
Question 3 (10 marks).

[Learning Goals:
• Prove semantic entailment by using a direct proof or a proof by contradiction.
• Disprove semantic entailment by giving a counterexample.]

Prove or disprove each of the given semantic entailments.

(a) \[
\left\{\left(\forall x \ P(x)\right) \lor \left(\forall x \ Q(x)\right)\right\} \vdash \left(\forall x \ (P(x) \lor Q(x))\right)
\]
(b) \[
\{(\forall x (P(x) \lor Q(x)))\} \vdash ((\forall x P(x)) \lor (\forall x Q(x)))
\]
Question 4 (8 marks).

[Learning Goals: Prove a conclusion from given premises using natural deduction inference rules.]

Give proofs in Natural Deduction showing each of the following. For full credit, use ONLY the basic rules of the Natural Deduction system (Huth and Ryan, top of p. 27, or Syntax and Semantics..., p. 4, from the Web page), plus ∀e, ∃i, ∀i and ∃e. You may receive partial credit if you use one of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or ¬¬i.

(a) \[
\{ (\forall x (P(x) \land Q(x))) \} \vdash ((\exists x P(x)) \land (\exists y Q(y)))
\]

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(b) \[
\emptyset \vdash ((\forall x P(x)) \rightarrow (\exists x P(x)))
\]

[4]
Question 5 (8 marks).
The proof of the Relevance Lemma from the slides was left as an exercise. One ingredient in that proof is the following claim.

**Claim 1** Let $t$ be any well-formed term of Predicate Logic. Let $I$ be any interpretation. Let $E_1$ and $E_2$ be any two environments such that $E_1(x) = E_2(x)$ for every variable symbol, $x$, that occurs in $t$. Then

$$t(I, E_1) = t(I, E_2).$$

Prove Claim 1 by structural induction on $t$. Be careful to lay out all parts of your induction argument explicitly.