Due Wednesday, Nov. 1, by 4:00pm, to Crowdmark.

All submitted work must be the student’s own.

Please note the following facts:

- From A05 onwards, all truth valuations, truth tables and valuation trees **MUST** use F/T, **NOT** 0/1 any longer.
- The end of the material covered by the mid-term exam will be the material covered on this assignment.

**Question 1** (12 marks).

*Learning Goal: Prove semantic entailment by using a direct proof or a proof by contradiction.*

Prove each of the given semantic entailments.

(a) \[
\{ (\forall x (\exists y (P(x) \lor Q(y))) ) \} \models (\exists y (\forall x (P(x) \lor Q(y))) )
\]
(b)

\[
\{ (\forall x (\forall y (P(y) \rightarrow Q(x))) ) \} \models (\exists y (P(y) \rightarrow (\forall x Q(x))))
\]
Question 2 (18 marks).

[Learning Goal: Disprove semantic entailment by giving a counterexample.]

Consider the following three Predicate logic sentences that state that (in all interpretations) a binary predicate symbol $P$ is reflexive, symmetric, and transitive:

$$\varphi_r = (\forall x P(x, x))$$
$$\varphi_s = (\forall x (\forall y (P(x, y) \rightarrow P(y, x))))$$
$$\varphi_t = (\forall x (\forall y (P(x, y) \rightarrow (P(y, z) \rightarrow P(x, z)))))$$

(a) Show that $\{\varphi_s, \varphi_r\} \not\models \varphi_t$.

(b) Show that $\{\varphi_s, \varphi_t\} \not\models \varphi_r$. 

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(c) Show that \( \{ \varphi_r, \varphi_t \} \not\subseteq \varphi_s \).
Question 3 (4 marks).

[Learning Goals: Describe the additional rules of inference for the two quantifiers.]

Explain what is wrong with the following Natural Deduction proof. I.e. identify any incorrect steps and explain why each is wrong.

1. \( (\forall x ((P(x) \land Q(x)) \rightarrow R(x))) \)  Premise
2. \( ((\exists x P(x)) \land (\exists x Q(x))) \)  Premise
3. \( (P(y) \land Q(y)), y \text{ fresh} \)  Assumption
4. \( ((P(y) \land Q(y)) \rightarrow R(y)) \)  \( \forall e: 1 \)
5. \( R(y) \)  \( \rightarrow e: 3,4 \)
6. \( (\exists x R(x)) \)  \( \forall i: 5 \)
7. \( (\exists x R(x)) \)  \( \exists e: 3-6 \)
Question 4 (8 marks).

[Learning Goals: Prove a conclusion from given premises using natural deduction inference rules.]

Give proofs in Natural Deduction showing each of the following. For full credit, use ONLY the basic rules of the Natural Deduction system (Huth and Ryan, top of p. 27, or Syntax and Semantics..., p. 4, from the Web page), plus ∀e, ∃i, ∀i and ∃e. You may receive partial credit if you use one of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or ¬¬i.

(a) \[
\left\{ \forall x \left( \forall y (P(y) \to Q(x)) \right) \right\} \vdash \left( \exists y (P(y) \to (\forall x Q(x))) \right)
\]

(b) \[
\left\{ (\exists x (P(x) \to Q(x)), (\forall y P(y)) \right\} \vdash (\exists x Q(x))
\]

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Question 5 (10 marks).

[Learning Goals: Prove semantic entailment by using a direct proof or a proof by contradiction. Prove a conclusion from given premises using natural deduction inference rules.]

(a) Prove that
\[ \emptyset \models \left( \exists x \ (R(x) \rightarrow (\forall y \ R(y))) \right), \]

i.e. show that the formula \( \exists x \ (R(x) \rightarrow (\forall y \ R(y))) \) is valid (i.e., satisfied by any interpretation and environment).
(b) Give a Natural Deduction proof of

$$\emptyset \vdash (\exists y (R(y) \rightarrow (\forall x R(x))))$$.

You may use derived rules—either ones from the text or notes or ones you prove yourself.

(Hint: You may consider using “$$((\forall x R(x)) \lor (\neg(\forall x R(x))))$$: L.E.M.” as the first line of your proof.)