Due Wednesday, Oct. 31, by 4:00pm, to Crowdmark.

All submitted work must be the student’s own.

Question 1 (5 marks).
[Learning Goals: Translate an argument to Predicate Logic. Prove a conclusion from given premises using Natural Deduction inference rules.]

In CS 136, you were subject to the following conditions on passing the course:

All students who passed CS 136 must have passed the weighted average of their assignments.

You hear some rumours in the hallowed halls of the University of Waterloo that:

There exists a student that passed CS 136 and that did not pass the weighted average of their assignments.

Translate the above premises into well-formed Predicate logic formulas. Then, give a Natural Deduction proof that the above can prove the following sentence (which you should also translate into Predicate logic):

All students who were in CS 136 received 100%.

For full credit, use ONLY the basic rules of the Natural Deduction system for Propositional logic ($\land i$, $\land e$, $\forall i$, $\forall e$, $\rightarrow i$, $\rightarrow e$, $\neg i$ (aka $\neg e$), $\bot e$, and $\neg\neg e$) and for Predicate logic ($\forall e$, $\exists i$, $\forall i$ and $\exists e$).
Question 2 (20 marks).

[Learning Goals: Prove a conclusion from given premises using Natural Deduction inference rules.]

Give proofs in Natural Deduction showing each of the following.

For full credit, use ONLY the basic rules of the Natural Deduction system for Propositional logic ($\land i, \land e, \lor i, \lor e, \rightarrow i, \rightarrow e, \neg i, \bot i$ (aka $\neg e$), $\bot e$, and $\neg \neg e$) and for Predicate logic ($\forall i, \exists i, \forall e$ and $\exists e$).

You may receive partial credit if you use one of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or $\neg \neg i$.

(a) 
\[
\left\{ \left( \neg \forall x \neg P(x) \right) \right\} \vdash_{ND} \exists x P(x)
\]

(b) 
\[
\left\{ \left( \neg \exists x \neg P(x) \right), \forall y Q(y) \right\} \vdash_{ND} \forall y \left( P(y) \land Q(y) \right)
\]
(c) \[
\{ (\exists z \ P(a, z)), (\exists x \ (P(a, x) \to Q(x))), (\neg (\exists x \ Q(x))) \} \vdash_{ND} \neg (\forall x (\forall y P(x, y)))
\]

(d) \[
\{ (\forall x (P(x) \to Q(a))) \} \vdash_{ND} \exists z ((\neg P(z)) \lor Q(z))
\]
Give a proof in Natural Deduction of the following. You may use any of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or $\neg\neg i$, without loss of credit.

\[
\begin{align*}
\left\{ \left( \forall x \left( P(x) \rightarrow \exists y \ Q(y, x) \right) \right) \right\} & \vdash_{\text{ND}} \left( \exists z \left( Q(z, a) \lor \left( \neg \forall x \ P(x) \right) \right) \right)
\end{align*}
\]
Question 4 (5 marks).

[Learning Goals: Prove a conclusion from given premises using Natural Deduction and Equality axioms.]

In this question, the subscript \( ND = \) refers to the rules of Natural Deduction for Predicate Logic, plus the axioms of equality \( EQ1 \) and \( EQ2 \) reproduced below, plus the symmetry and transitivity of equality.

**EQ1.** \((\forall u (u = u))\) is an axiom.

**EQ2.** For each formula \( \varphi \) and variable \( t \),

\[
(\forall r \left( (r = s) \rightarrow (\varphi[r/t] \rightarrow \varphi[s/t]) \right))
\]

is an axiom.

Let \( \{w, x, y, z\} \) be variable symbols. Let \( R \) be a unary relation symbol.

Prove the following: For any well-formed formula \( \varphi \) of Predicate Logic,

\[
\emptyset \vdash_{ND} (\forall x(\forall y(((\varphi \rightarrow R(x)) \land (x = y)) \rightarrow (\varphi \rightarrow R(y)))).
\]

Hint: If you substitute something for a variable, don’t forget to do the substitution with \( \varphi \)!