Question 1 (10 marks).

[Learning Goal: Prove results using soundness and completeness of Natural Deduction for Predicate Logic.]

(a) Show that no Natural Deduction proof exists to witness

\[ \{(\exists x (P(x) \rightarrow Q(x))), (\exists y P(y))\} \vdash (\exists x Q(x)). \]
(b) Let \( \Sigma \) be a set of Predicate formulæ. Let \( \varphi, \psi \) be Predicate formulæ. Suppose that \( \Sigma \cup \{ \psi \} \models \varphi \). Does it follow that \( \Sigma \vdash \varphi \)? If it does follow, then give a proof. If it does not follow, then give an example of \( \Sigma, \varphi \) and \( \psi \) such that \( \Sigma \cup \{ \psi \} \models \varphi \) and \( \Sigma \nvdash \varphi \), and explain why your example is correct.
Question 2 (5 marks).

[Learning Goal: Prove results using Natural Deduction with Equality.]

In this question, the subscript \( \text{ND} \) refers to the rules of Natural Deduction for Predicate Logic, and the subscript \( \text{ND=} \) refers to the rules of Natural Deduction for Predicate Logic, plus the axioms of equality \( \text{EQ1} \) and \( \text{EQ2} \) reproduced below, plus the symmetry and transitivity of equality.

\( \text{EQ1.} \) \((\forall x \ (x = x))\) is an axiom.

\( \text{EQ2.} \) For each formula \( \varphi \) and variable \( z \),

\[
(\forall x \ (\forall y \ ((x = y) \rightarrow (\varphi[x/z] \rightarrow \varphi[y/z])))
\]

is an axiom.

Give a full proof of the following sentence in Natural Deduction, where \( f \) is a function symbol and \( = \) is equality. Note that induction is not required to prove this result.

\[
\emptyset \vdash_{\text{ND=}} (\forall x (\forall y (\exists z (f(x, y) = z)))
\]

When you use a rule that replaces a variable by a term, or a term by a variable, identify the term in the explanation. (This may help you; it definitely helps the marker.)
Question 3 (8 marks).

[Learning Goal: Prove results using Peano Arithmetic.]

For the following, give proofs using the axioms of Peano Arithmetic. You may use the derived rules for equality, such as EqSubs and EqTrans(·).

(a) The predicate “$x$ is greater than or equal to $y$” is defined by the formula $(\exists z (x = z + y))$.

Give a formal proof that, if $x$ is greater than or equal to $y$ then the successor of $x$ is greater than or equal to the successor of $y$. That is, provide a proof for

$$\{ (\exists z (x = z + y)) \} \vdash_{PA} (\exists z (s(x) = z + s(y))) .$$
(b) The predicate “$z$ is even” can be defined by the formula $(\exists u (u + u = z))$. Give a formal proof that, if $x$ is even, then $s(s(x))$ is also even. That is, provide a proof for

$$\{ (\exists u (u + u = z)) \} \vdash_{PA} (\exists u (u + u = s(s(z))))$$.
Question 4 (12 marks).

[Learning Goal: Prove results using Peano Arithmetic.]

Give a formal proof of right-distributivity in Peano Arithmetic; that is, give a proof for

$$\emptyset \vdash_{PA} (\forall z ((x + y) \times z) = ((x \times z) + (y \times z)))$$

Remark: Proving this fact for a free $x$ and a free $y$ provides the content of the subproof that would ultimately permit us to prove the result for all $x$ and for all $y$.

In your proof, you may freely use

- any of the basic or derived rules of Natural Deduction,
- any of the properties of equality from the course overheads (symmetry, transitivity, EqSubs and EqTrans$(k)$), and/or
- the commutativity and associativity of $+,$ as expressed by

$$\emptyset \vdash_{PA} \left(\forall u \left(\forall v ((u + v) = (v + u))\right)\right) \quad \text{and} \quad \emptyset \vdash_{PA} \left(\forall u \left(\forall v \left(\forall w (((u + v) + w) = (u + (v + w)))\right)\right)\right).$$

If you wish, you may combine uses of $\forall$-elimination with other rules.

For simplicity of marking, lay out your solution as follows.

(a) Give the two formulas to be proven as the base case and the inductive case, respectively.

(b) Prove the base case.

(c) Prove the inductive case.

(d) Complete the proof, using the base and inductive cases as lemmas.

Identify the parts as given, but submit all parts together. As always, your submitted solution may have any number of pages.