Due Wednesday, Jul. 4, by 4:00pm, to Crowdmark.
All submitted work must be the student’s own.

Question 1 (10 marks).

[Learning Goals: Prove a conclusion from given premises using Natural Deduction inference rules.]

Give proofs in Natural Deduction showing each of the following.
For full credit, use ONLY the basic rules of the Natural Deduction system for Propositional logic ($\land i$, $\land e$, $\lor i$, $\lor e$, $\to i$, $\neg i$, $\neg e$, $\bot e$, and $\neg \neg e$) and for Predicate logic ($\forall e$, $\exists i$, $\forall i$ and $\exists e$).
You may receive partial credit if you use one of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or $\neg \neg i$.

(a) 
\[
\{ (\exists x (P(x) \rightarrow Q(x))), (\forall y P(y)) \} \vdash (\exists x Q(x))
\]
(b) \[
\left\{ \forall x \left( \forall y (P(y) \rightarrow Q(x)) \right) \right\} \vdash (\exists y (P(y) \rightarrow (\forall x Q(x))))
\]

[4]

(c) \[
\left\{ \neg (\forall x P(x)) \right\} \vdash (\exists x \ (\neg P(x)))
\]

[2]
Question 2 (20 marks).

[Learning Goals: Prove a conclusion from given premises using Natural Deduction inference rules.]

Give proofs in Natural Deduction showing each of the following. You may use any of the derived rules MT (modus tollens), PBC (proof by contradiction), LEM (law of excluded middle), or ¬¬i, without loss of credit. You may use any parts of question 1 even if you did not complete the proofs.

(a) \[ \emptyset \vdash (\exists y (R(y) \rightarrow (\forall x R(x))). \]
(b) \[\{(\forall x \, P(x))\} \vdash (\exists y \, (\forall x \, (P(x) \lor Q(y))))\]

(c) \[\{(\neg (\forall x \, P(x))), (\forall x \, (\exists y \, (P(x) \lor Q(y))))\} \vdash (\exists y \, Q(y))\]
\[ \left\{ (\forall x \exists y (P(x) \lor Q(y))) \right\} \vdash (\exists y (\forall x (P(x) \lor Q(y)))) \]

Hint 1: You may consider using "\((\forall x P(x)) \lor (\neg(\forall x P(x)))\): L.E.M." as the first line of your proof.

Hint 2: You may want to use your solutions to Q2b and Q2c.
Question 3 (3 marks).

[Learning Goal: Prove that an inference rule is sound.]

Suppose that $\alpha$ and $\beta$ are well-formed Predicate formulas, $t$ is a term and $x$ is a variable. Consider the following inference rule

$$
\frac{(\forall x (\alpha \rightarrow \beta)) \alpha[t/x]}{\beta[t/x]} (\forall e^*)
$$

Prove that the $\forall e^*$ inference rule is sound, i.e. prove that the following semantic entailment holds.

$$
\{(\forall x (\alpha \rightarrow \beta)), \alpha[t/x]\} \models \beta[t/x].
$$

You must use the definition of semantic entailment to write your proof. Do not use any other technique such as truth tables, valuation trees, logical equivalence, Natural Deduction, soundness, or completeness.

You may use Lemmas 1 and 2 below in your proof.

**Lemma 1** Let $t$ be a Predicate term. Let $I$ be an interpretation. Let $E$ be an environment. Then we have that

$$
t(I, E) \in \text{dom}(I).
$$

for every interpretation $I$ and environment $E$.

**Lemma 2** Let $\alpha$ be a well-formed Predicate formula. Let $t$ be a Predicate term. Let $I$ and $E$ be an interpretation and environment. Let $x$ be a variable. Then we have that

$$
\alpha[t/x]^{(I, E)} = \alpha^{(I, E[x \mapsto t(I, E)])}.
$$

for every interpretation $I$ and environment $E$. 
Question 4 (4 marks).
[Learning Goal: Prove that an inference rule is not sound.]

Suppose that \( \alpha \) is a well-formed Predicate formula, \( t \) is a term and \( x \) is a variable. Consider the following inference rule.

\[
\frac{(\exists x \ \alpha)}{\alpha[t/x]} (\exists^*)
\]

Prove that the \( \exists^* \) inference rule is NOT sound, i.e. **prove the following statement**

\[\{(\exists x \ \alpha)\} \not\models \alpha[t/x].\]

You **must use the definition of semantic entailment** to write your proof. Do not use any other technique such as truth tables, valuation trees, logical equivalence, Natural Deduction, soundness, or completeness.

You may use Lemmas 1 and 2 in your proof.
Question 5 (4 marks).

[Learning Goal: Prove results using soundness and completeness of Natural Deduction for Predicate Logic.]
Suppose that $\alpha$ and $\beta$ are well-formed Predicate formulas, and $\Sigma$ is a set of well-formed Predicate formulas.
Suppose that $\Sigma \cup \{\beta\} \models \alpha$. Does it follow that $\Sigma \vdash \alpha$?
If it does follow, then give a proof. If it does not follow, then give an example of $\Sigma, \alpha$ and $\beta$ such that $\Sigma \cup \{\beta\} \models \alpha$ and $\Sigma \nvdash \alpha$, and explain why your example is correct.