Due Wednesday, Nov. 28, by 4:00pm, to Crowdmark.

All submitted work must be the student's own.

This is "page 0" of the assignment. Crowdmark has a place for submissions to it but since no marks are allocated, you may leave it empty. Submissions can start with question 1.

The code at right might be used as part of an "insertion sort". It assumes that an array A (with elements $A[1], A[2], \ldots, A[n]$) is sorted in increasing order up to position $n-1$, and it moves element $A[n]$ into its correct place.

The condition $\big((i \geq 1) \wedge \big(S(1,(i-1)) \wedge S(i,n)\big)\big)$ is suggested as a loop invariant. It uses the symbol $S(x,y)$ to denote that array A is sorted from position x to position y; that is, $S(x,y)$ is the formula

$$\Big(\forall j \big(((x < j) \wedge (j \leq y)) \rightarrow (A[(j-1)] \leq A[j])\big)\Big) \ .$$

Note that if $(y \leq x)$, then $S(x,y)$ always holds, regardless of the content of A.

$\{\!|\ ((n > 0) \wedge S(1,(n-1)))\ |\!\}$
```
i = n ;
```
$\{\!|\ \big((i \geq 1) \wedge (S(1,(i-1)) \wedge S(i,n))\big)\ |\!\}$
```
while ( i > 1 ) {
    if ( A[i] < A[i-1] ) {
        t = A[i] ;
        A[i] = A[i-1] ;
        A[i-1] = t ;
    }
    i = i - 1 ;
}
```
$\{\!|\ S(1,n)\ |\!\}$

In the course of doing the assignment, you will discover that the suggested invariant is actually insufficient to prove correctness. Once you determine why, you will have a chance to correct it.

Overview of questions

Except for question 2, clarification appears on the question page.

Question 1: Some preliminary work to investigate the formulas "$S(x,y)$" and their behaviour under substitution.

Question 2: Annotations. Use the given template, NOT a substitute. Leave entries blank as appropriate.

(a) Using the suggested loop invariant, add annotations to the template for the `while`-loop and the first condition following the `if` statement, for proving partial correctness.

(b) For each assignment statement, determine the pre-condition required for the assignment rule. Apply any appropriate simplifications from Question 1, and enter the *simplified* formulas into the template.

Question 3: Justification of "implied" formulas — insofar as possible.

Question 4: Modification of the invariant and of the justifications.

Question 5: Proving total correctness.

Question 1 (4 marks).
As preparation for the annotation, show that each of the following equivalences holds, for every k and every z.

(a) If $((1 \leq \ell) \wedge (\ell < k))$, then $S(1, \ell)[A\{k \leftarrow z\}/A]$ is equivalent to $S(1, \ell)$.

(b) If $((k < \ell) \wedge (\ell \leq n))$, then $S(\ell, n)[A\{k \leftarrow z\}/A]$ is equivalent to $S(\ell, n)$.

(c) If $(k < n)$, then $S(k, n)[A\{k \leftarrow z\}/A]$ is equivalent to $\big((z \leq A[(k+1)]) \wedge S((k+1), n)\big)$.

Question 2 (6 marks).
Fill in the template with appropriate annotations. For more explanation, see the introduction page. (If you have a medical condition that renders entry by hand into the template difficult or impossible, contact your instructor regarding an alternative.)

$(\!| \, ((n > 0) \wedge S(1, (n-1))) \, |\!)$

```
i = n ;
```

$(\!| \, ((i \geq 1) \wedge (S(1, (i-1)) \wedge S(i, n))) \, |\!)$

```
while ( i > 1 ) {
    if ( A[i] < A[i-1] ) {
        t = A[i] ;
        A[i] = A[i-1] ;
        A[i-1] = t ;
    }
    i = i - 1 ;
}
```

$(\!| \, S(1, n) \, |\!)$

Question 3 (6 marks).

(a) Justify as many of the implied conditions as you can. For "implied" proofs, use ordinary arithmetic laws. (As you would have in MATH 135.)

(b) Specify which implied condition has no proof. Explain precisely why it fails.

Question 4 (6 marks).

(a) Give a modified loop invariant that would suffice to complete the proof of partial correctness. Describe how your modified invariant affects the "implied" conditions and their proofs, in the previous question.

(b) Justify the formerly failing condition.

Question 5 (6 marks).
Show that the program is totally correct.
Assume partial correctness, whether you proved it or not.