Study Exercises

Exercise 1.
Let Σ be a set of well-formed Propositional formulæ and let φ be a well-formed Propositional formula. Prove that if $\varphi \in \Sigma$, then $\Sigma \vDash \varphi$.

Exercise 2.
Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be well-formed Propositional formulæ . Prove by induction on n that the following semantic entailment holds for any Natural number, n:

$$\{\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n\} \vDash (\cdots((\varphi_0 \wedge \varphi_1) \wedge \varphi_2) \wedge \cdots) \wedge \varphi_n)$$

Exercise 3.
Give proofs of the following, in Natural Deduction for Propositional logic. Try to restrict yourself to only the basic rules. (Note: the laws of algebra are **not** proof rules. These are semantic. Do **not** use them in a formal Natural Deduction proof.)

(a) $\{(\varphi \vee \eta)\} \vdash (\neg((\neg\varphi) \wedge (\neg\eta)))$.

(b) $\{(\neg(\varphi \wedge \eta))\} \vdash ((\neg\varphi) \vee (\neg\eta))$.

(c) $\{((\varphi \wedge \eta) \vee (\varphi \wedge \zeta))\} \vdash (\varphi \wedge (\eta \vee \zeta))$.

(d) For additional practice, see any of Exercises 1.2.1–3 in Huth and Ryan, pp. 78–80.

Exercise 4.
Prove that for any set of Propositional formulæ Σ and any Propositional variables p and q, if $\Sigma \vdash p$ then $\Sigma \vdash ((\neg p) \to q)$.

Exercise 5.
Let α_1, α_2 and β be Propositional formulæ , where $\emptyset \vdash (\alpha_1 \to \alpha_2)$. Let Σ be a set of Propositional formulæ . Show that if $\Sigma \cup \{\alpha_2\} \vdash \beta$ then also $\Sigma \cup \{\alpha_1\} \vdash \beta$.

Exercise 6.
Let Σ be any set of well-formed Propositional formulæ .

Definition: We say that Σ is **inconsistent** if it proves a contradiction, i.e. if $\Sigma \vdash \bot$. Σ is **consistent** if it is not inconsistent, i.e. if $\Sigma \nvdash \bot$.

(a) Let Σ be consistent. Let φ be a well-formed Propositional formula. Prove that **at most one of** $\Sigma \vdash \varphi$ and $\Sigma \vdash (\neg\varphi)$ can hold.

(b) Prove that intersections of consistent sets are consistent, but that unions of consistent sets need not be consistent.

(c) Is the empty set, $\emptyset$, consistent or inconsistent? Prove your answer.

(d) For Natural Deduction proofs, prove that Σ is inconsistent if and only if Σ is unsatisfiable.