Exercise 1.
Let $\Sigma$ be a set of well-formed Propositional formulæ and let $\varphi$ be a well-formed Propositional formula. Prove that if $\varphi \in \Sigma$, then $\Sigma \models \varphi$.

Exercise 2.
Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ be well-formed Propositional formulæ. Prove by induction on $n$ that the semantic entailment
\[
\{\varphi_0, \varphi_1, \varphi_2, \ldots, \varphi_n\} \models (\cdots ((\varphi_0 \land \varphi_1) \land \varphi_2) \land \cdots) \land \varphi_n
\]
holds for any Natural number, $n$.

Exercise 3.
Give proofs of the following, in Natural Deduction for Propositional logic. If you find proofs that use one or more derived rules, use them as hints to find a proof using only basic rules. (Note: the laws of Boolean algebra are not proof rules. Do not use them in a formal Natural Deduction proof.)

(a) $\{((\varphi \lor \eta)) \vdash (\neg(((\neg \varphi) \land (\neg \eta)))$.
(b) $\{(\neg(\varphi \land \eta)) \vdash ((\neg \varphi) \lor (\neg \eta))$.
(c) $\{((\varphi \land \eta) \lor (\varphi \land \zeta)) \vdash (\varphi \land (\eta \lor \zeta))$.
(d) For additional practice, see any of Exercises 1.2.1–3 in Huth and Ryan, pp. 78–80. (You won’t have time to do all of them!)

Exercise 4.
Prove that for any set of Propositional formulæ $\Sigma$ and any Propositional variables $p$ and $q$, if $\Sigma \vdash p$ then $\Sigma \vdash ((\neg p) \rightarrow q)$.

Exercise 5.
Let $\alpha_1, \alpha_2$ and $\beta$ be Propositional formulæ, where $\emptyset \vdash (\alpha_1 \rightarrow \alpha_2)$. Let $\Sigma$ be a set of Propositional formulæ. Show that if $\Sigma \cup \{\alpha_2\} \vdash \beta$ then also $\Sigma \cup \{\alpha_1\} \vdash \beta$.

Exercise 6.
Let $\Sigma$ be any set of well-formed Propositional formulæ.

Definition: We say that $\Sigma$ is inconsistent if it proves a contradiction, i.e. if $\Sigma \vdash \bot$. $\Sigma$ is consistent if it is not inconsistent, i.e. if $\Sigma \not\vdash \bot$.

(a) Let $\Sigma$ be consistent. Let $\varphi$ be a well-formed Propositional formula. Prove that at most one of $\Sigma \vdash \varphi$ and $\Sigma \vdash (\neg \varphi)$ can hold.

(b) Prove that intersections of consistent sets are consistent, but that unions of consistent sets need not be consistent.
(c) Is the empty set, $\emptyset$, consistent or inconsistent? Prove your answer.