Study Exercises

Exercise 1.
Huth and Ryan, Exercises 2.1, Problem 1

Exercise 2.
Huth and Ryan, Exercises 2.1, Problem 3

Exercise 3.
Huth and Ryan, Exercises 2.2, Problem 1

Exercise 4.
Huth and Ryan, Exercises 2.2, Problem 3

Exercise 5.
For this question, we use the symbols '+', '·' and '=' with their normal mathematical meanings: addition, multiplication and equality, respectively. Note that '0', '1', etc., are NOT included as symbols.

(a) Give formulæ for each of the following. Assume that the domain is $\mathbb{N}$, the **natural numbers**.

i. "x is the number one."

ii. "x is an even number."

iii. "Every number is less that its square."

iv. "x is composite number." (Try it first without worrying that neither zero nor one is technically composite. Then check whether your formula handles them correctly, and fix it if not.)

(b) Consider the formula

$$(\forall y\,(\forall y\,(x = y \cdot z \to (y = x \vee z = x))))$$

which might be written to mean "x is a prime number".

For each of the domains below, explain what it actually means; that is, describe the set of x in the domain that make the formula true. If the set is finite, list its elements; if the set is infinite, describe in one English phrase or sentence the elements of the set.

i. Domain $\mathbb{N}$, the **natural numbers** (including zero).

ii. Domain $\mathbb{Z}$, the **integers**.

iii. Domain $\mathbb{Q}$, the **rational numbers**.

Exercise 6.
On the overheads, you saw the example formula

$$\left(\forall\varepsilon\left(\varepsilon>0\to\left(\exists\delta\left(\delta>0\wedge\left(\left(\forall y\ |x-y|<\delta\right)\to|f(x)-f(y)|<\varepsilon\right)\right)\right)\right)\right),$$

which, using the usual interpretation of the real numbers, expresses that function f is continuous at the point x. To express that f is continuous everywhere, we can add "$\forall x$" in front of that formula. What about "f is uniformly continuous"? Consider the following formulæ – which one(s), if any, express uniform continuity? What do the others express?

(a) $(\forall\varepsilon\,(\forall x\left(\varepsilon>0\to(\exists\delta\,(\delta>0\wedge(\forall y\,(|x-y|<\delta\to|f(x)-f(y)|<\varepsilon))))\right)))$

(b) $(\forall\varepsilon\left(\varepsilon>0\to(\forall x\,(\exists\delta\,(\delta>0\wedge(\forall y\,(|x-y|<\delta\to|f(x)-f(y)|<\varepsilon)))))\right))$

(c) $(\forall\varepsilon\left(\varepsilon>0\to(\exists\delta\,(\forall x\,(\delta>0\wedge(\forall y\,(|x-y|<\delta\to|f(x)-f(y)|<\varepsilon)))))\right))$

(d) $(\forall\varepsilon\left(\varepsilon>0\to(\exists\delta\,(\delta>0\wedge(\forall x\,(\forall y\,(|x-y|<\delta\to|f(x)-f(y)|<\varepsilon)))))\right))$

Pay particular attention to the quantifiers and their placement in the formulas.

Exercise 7.
For each of the formulæ listed below, give an interpretation $\mathcal{I}$ in which it is true and an interpretation $\mathcal{J}$ in which it is false. (Symbols a and b are constant symbols.)

(a) $\left(\forall x\,((P(x)\wedge Q(x,a))\to Q(x,b))\right)$

(b) $\left(\forall x\left(\exists y\,(P(y,a)\to(\neg Q(x,y)))\right)\right)$

(c) $\left(\exists x\,(\forall y\,(P(f(a),x)\wedge Q(g(x,x),y)))\right)$

Exercise 8.
Show that a Predicate formula φ is a valid formula if and only if the formula $(\forall x\ \varphi)$ is a valid formula.

Exercise 9.
Let Σ be a set of well-formed Predicate fomulæ and let φ and α be well-formed Predicate fomulæ. Let $\mathcal{I}$ be any interpretation. Let E be any environment. Let $\emptyset$ denote the **empty set**.

(a) Suppose that $\Sigma\models\varphi$. Does it follow that $\Sigma\cup\{\alpha\}\models\varphi$? Give a proof or a counterexample.

(b) Suppose that $\Sigma\cup\{\alpha\}\models\varphi$. Does it follow that $\Sigma\models\varphi$? Give a proof or a counterexample.

(c) Prove that $\mathcal{I}\models_E\emptyset$.

(d) Prove that if $\emptyset\models\varphi$, it follows that φ is **valid**.

(e) Give an exmple to show that the converse of the statement in part 9d is false. In detail, give an example of a valid Predicate formula φ and a **non-empty** set Σ such that $\Sigma\models\varphi$ holds.

Exercise 10.
Prove the **Relevance Lemma**:
Let α be a Predicate formula, let $\mathcal{I}$ be an interpretation, and let E_1 and E_2 be two environments

such that

$$E_1(x) = E_2(x) \text{ for every } x \text{ that occurs free in } \alpha.$$

Then

$$\boldsymbol{\mathcal{I}} \vDash_{E_1} \alpha \text{ if and only if } \boldsymbol{\mathcal{I}} \vDash_{E_2} \alpha \ .$$

Exercise 11.
For additional practice, see any of Exercises 2.4.7–12 in Huth and Ryan, p. 164. You won't have time to do all of them! N.B. In Exercise 2.4.11, replace "consistent" with "satisfiable".