Exercise 1.
Prove the following via Natural Deduction.

(a) “Adding a non-zero number yields a different value”; i.e., \( \emptyset \vdash_{PA} \left( \forall x \left( \forall y \left( -\left(s(x) + y = y\right)\right)\right) \right) \).

(b) “The product of non-zero numbers is non-zero”; i.e., \( \emptyset \vdash_{PA} \left( \forall x \left( \forall y \left( \exists z \left( s(x) \times s(y) = s(z)\right)\right)\right) \).

(c) “The number 2 is prime”, in the form \( \emptyset \vdash_{PA} \left( \forall y \left( \forall z \left( -\left(s(s(y)) \times s(s(z)) = s(s(0))\right)\right)\right) \).

Hint for (c): You may find parts (a) and (b) helpful.

Exercise 2.
Let “\( \alpha > \beta \)” be an abbreviation for \( \left( \exists u \left( \alpha = (\beta + s(u))\right)\right) \) (where \( u \) is any fresh variable). Describe carefully how you would prove (in Peano Arithmetic) the formula
\[
\left( \forall x \left( s(s(x)) \times s(s(x)) > s(s(x))\right) \right).
\]

For example, you might:

(a) Write down the actual formula that the above abbreviates.

(b) Identify an appropriate instance of PA7, including the formulas for the base case and the inductive case of the proof.

(c) If a full proof would require any basic properties of arithmetic, indentify them—for example, associativity of + and/or \( \times \), distributivity, etc. Explain why the property is required.

(d) You may, if you wish, write out any part of the formal proof, in order to determine how or why it works. (Remember, this is a study question—do what you require to understand how to do the proof, but skip anything that’s simply “busy work” for you.)

Exercise 3.
For each of the following properties or functions, give a Predicate formula that defines the corresponding relation. In the case of a property, give a formal proof (in PA) that some natural numbers satisfy the relation and some do not. In the case of a function, prove that the formula you give actually does define a function.

(a) “\( x \) is the sum of two squares.”

(b) “\( x \) is a power of a prime”.

(c) The function “monus”, defined as
\[
\text{monus}(x, y) = \begin{cases} 
0 & \text{if } x < y \\
 x - y & \text{if } x \geq y 
\end{cases}
\]