Study Exercises

Exercise 1.
Prove the following using Natural Deduction.

(a) \( \{ (\forall x (\forall y P(x, y))) \} \vdash (\forall y (\forall x P(x, y))) \).

(b) \( \{ (\forall x ((\neg P(x)) \land Q(x))) \} \vdash (\forall x (P(x) \rightarrow Q(x))) \).

(c) \( \{ (\forall x (P(x) \land Q(x))) \} \vdash (\forall x (P(x) \rightarrow Q(x))) \).

(d) \( \{ (\forall x (P(x) \land Q(x))) \} \vdash ((\forall x P(x)) \land (\forall x Q(x))) \).

(e) \{ (\forall x (P(x)) \lor (\forall x Q(x))) \} \vdash (\forall x (P(x) \lor Q(x))) \).

(f) \{ (\forall x (P(x) \rightarrow Q(x))) \} \vdash ((\forall x (\neg Q(x))) \rightarrow (\forall x (\neg P(x)))) \).

(g) \{ (\forall x (\forall y (R(x, y) \rightarrow R(y, x)))) \} \vdash (\forall x (\forall y (R(y, x) \rightarrow R(x, y)))) \).

(h) \{ (\forall x (\forall y (\forall z ((R(x, y) \land R(x, z)) \rightarrow R(y, z))))), (\forall x (R(x, x))) \} \vdash (\forall x (\forall y (R(x, y) \rightarrow R(y, z)))) \).

(i) \{ (\forall x (\forall y (\forall z ((R(x, y) \land R(y, z)) \rightarrow R(x, z))))), (\forall x (\neg R(x, x))) \} \vdash (\forall x (\forall y (R(x, y) \rightarrow (\neg R(y, z)))) \).

Exercise 2.
Prove the following using Natural Deduction.

(a) \{ (\exists x (P(x) \rightarrow Q(x))), (\forall y P(y)) \} \vdash (\exists x Q(x))

(b) \{ (\exists y (\exists x P(x, y))) \} \vdash (\exists y (\exists x P(x, y)))

(c) \{ (\exists x ((\neg P(x)) \land (\neg Q(x)))) \} \vdash (\exists x (\neg (P(x) \land Q(x))))

(d) \{ (\exists x ((\neg P(x)) \lor Q(x))) \} \vdash (\exists x (\neg (P(x) \land (\neg Q(x))))

(e) \{ (\exists x (P(x) \land Q(x))) \} \vdash ((\exists x P(x)) \land (\exists x Q(x)))

(f) \{ ((\exists x P(x)) \land (\exists x Q(x))) \} \vdash (\exists x (P(x) \lor Q(x)))

Exercise 3.
Prove the following using Natural Deduction.
Exercise 4.

(a) Prove that the following inference rule is sound.

\[
\frac{\forall x (\alpha \to \beta) \quad \alpha[t/x]}{\beta[t/x]} \quad \forall e_1
\]

where \(\alpha\) and \(\beta\) are predicate formulas, \(t\) is a predicate term, and \(x\) is a variable.

(b) Prove that the following inference rule is sound.

\[
\frac{\forall x (\alpha \to \beta) \quad (\neg \beta[t/x])}{\neg \alpha[t/x]} \quad \forall e_2
\]

where \(\alpha\) and \(\beta\) are predicate formulas, \(t\) is a predicate term, and \(x\) is a variable.

(c) Prove that the following inference rule is NOT sound.

\[
\frac{\forall x (\alpha \to \beta) \quad \beta[t/x]}{\alpha[t/x]} \quad \forall e_3
\]

where \(\alpha\) and \(\beta\) are predicate formulas, \(t\) is a predicate term, and \(x\) is a variable.

(d) Prove that the following inference rule is NOT sound.

\[
\frac{\forall x (\alpha \to \beta) \quad (\neg \alpha[t/x])}{\neg \beta[t/x]} \quad \forall e_4
\]

where \(\alpha\) and \(\beta\) are predicate formulas, \(t\) is a predicate term, and \(x\) is a variable.

(e) Let \(\Sigma\) be a set of Predicate formulas and let \(\alpha\) be a Predicate formula.

If \(\Sigma \vdash \alpha\), then \(\{\Sigma, (\neg \alpha)\}\) is unsatisfiable.
(f) Let $\Sigma$ be a set of Predicate formulas and let $\alpha$ be a Predicate formula.

If $\{\Sigma, (\neg \alpha)\}$ is unsatisfiable, then $\Sigma \vdash \alpha$. 